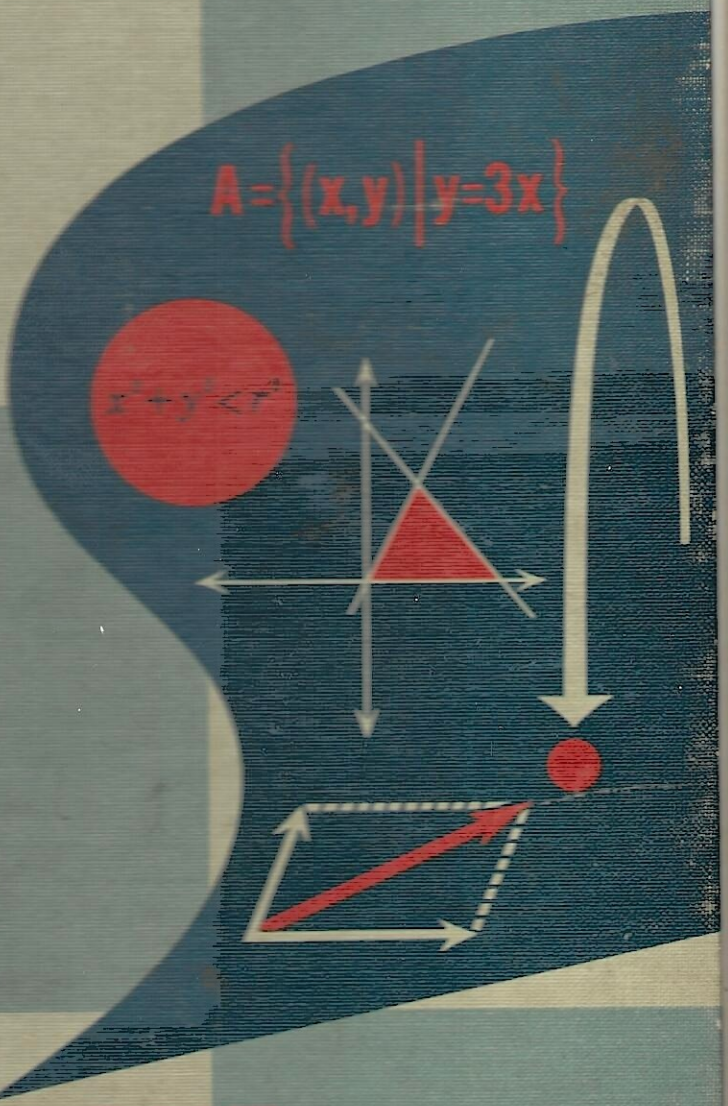
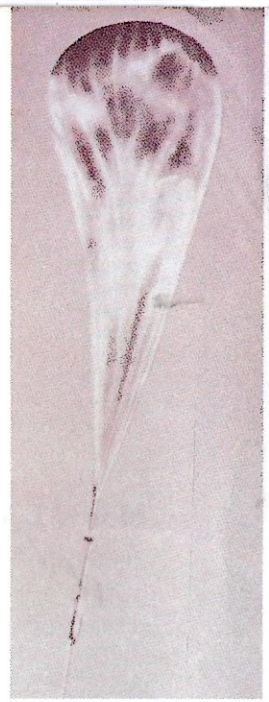
MODERN MATHEMATICS



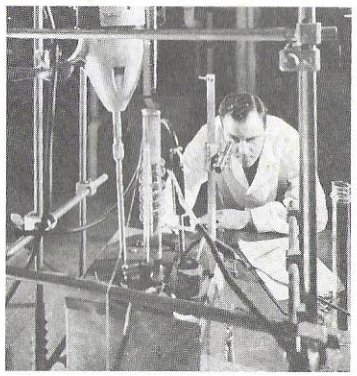
ALGEBRA ONE

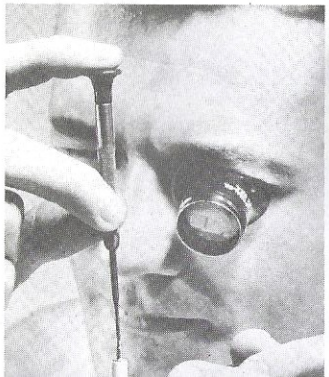
Part 3 of 3

MODERN MATHEMATICS *Algebra One*



Thus,
$1+4+7+\cdots+(3k-2)+(3k+1)$
$=\frac{1}{2}k(3k-1)+(3k+1)$
$=\frac{3}{2}k^2-\frac{1}{2}k+3k+1$
$=\frac{3}{2}k^2+\frac{5}{2}k+1$
$=\frac{1}{2}(3k^2+5k+2)$
$=\frac{1}{2}(k+1)(3k+2)$





MODERN MATHEMATICS

Algebra One

MYRON F. ROSSKOPF
Teachers College, Columbia University

ROBERT L. MORTON
Ohio University

JOSEPH R. HOOTEN
Laboratory School, The Florida State University

HARRY SITOMER
Polytechnic Institute of Brooklyn, New York

STEPHEN S. WILLOUGHBY
University of Wisconsin, Madison, Wisconsin

GLENN A. GILBERT
Boulder High School, Boulder, Colorado

SILVER BURDETT COMPANY

Morristown, N. J. Chicago San Francisco Dallas Atlanta

Contents

9. Quadratic Equations and Inequalities 290

Solving Quadratic Equations
Investigating $a^2 = b^2$ • Special Quadratic Equations • Solution by Factoring • Completing the Square • The Quadratic Formula

Solving Quadratic Inequalities
Principles for $a^2 < b^2$ and $a^2 > b^2$ • Completing the Square

Quadratic Functions
The Rule $y = Ax^2 + Bx + C$ • Lowest and Highest Points • Domain and Range • Using Graphs to Solve Equations and Inequalities

Problem Solving

10. Geometry 320

Coordinate Geometry
Line Segments and Midpoints • The Distance Formula • Equation of a Line • Collinear Points • Parallel and Perpendicular Lines • Algebraic Proofs • Equation of a Circle

Structure of a Mathematical System
Definitions • Undefined Words • Axioms • Theorems and Proof

11. Trigonometry 351

The Trigonometric Functions
Angle in Standard Position • Tangent Function • Sine Function • Snell's Law • Cosine Function

Vectors
Ordered Pairs as Vectors • Vector Addition • Vector Components

12. Elements of Probability 375

Introductory Ideas
Measuring Probability • Sample Spaces • Truth Set of an Event • Probability Trees

Extending the Use of Probability
Probability Symbol • Simple Conjunctions • Simple Disjunctions • Probability in Science

Extra Practice 396

Challenge 411

Tables 426

Preface

TEACHING of elementary algebra today must reflect recent developments in the realm of mathematics. Clearly, one aim of an algebra course is to acquaint the student with the algebra that is used functionally by present-day mathematicians. Therefore, the course must include topics not previously in the elementary algebra curriculum, for example: set theory; structure of number systems; inequalities as well as equations; the function concept; and greater attention to the important role of the coordinate plane. Needless to say, the development of these topics in a beginning course in algebra should be appropriate to the mathematical maturity of the student.

It should be emphasized that topics of modern algebra are not "substitutions" for topics of traditional algebra. Rather, modern topics provide a more meaningful way of developing understandings basic to the traditional topics.

Of crucial importance in the course is the spirit and technique used to bring the content to the student. By stressing individual inquiry and participation, and by leading the student through a carefully planned series of discoveries, teacher and text can provide important benefits for the student. These benefits include an *interest* in mathematics; an *understanding* of the foundations of mathematics; and *manipulative skills* based on understanding.

Algebra is not simply a small mathematical world in itself. It has a far-reaching role throughout mathematics, including the important areas of geometry, trigonometry, and analytic geometry. It is an important function of an elementary algebra course to communicate an awareness of this role to the student.

It is worth observing that modern algebra is not "difficult" algebra, suitable only for the better student. Through a heightened emphasis on meaning and structure, students of average or less-than-average ability can benefit as much as the superior student.

MODERN MATHEMATICS *Algebra One* is carefully planned to reflect the algebra course described above. It is the belief of the authors that teacher and student alike will find this course in modern algebra stimulating, meaningful, and an effective preparation for subsequent courses in mathematics.

CONTINUING FROM

PART TWO

OF THIS BOOK

9 Quadratic Equations and Inequalities

EQUATIONS

Suppose the height of a projectile fired directly upward is given by the formula $h = 2720t - 16t^2$, where h is the height (in feet) and t is the time (in seconds) after firing. To find the number of seconds before the projectile returns to earth, we might write $0 = 2720t - 16t^2$ and then solve for the value or values of t that satisfy this sentence. Will the sentence be true if t is replaced by 0? Do you think that this represents the time the projectile will return to earth? If not, what does it represent?

Is the sentence true if t is replaced by 170? Then, 170 seconds after firing, the projectile will come to earth.

Examine the equation $0 = 2720t - 16t^2$. One member of this equation is a polynomial in which one of the terms has a variable raised to the second power. We call the equation $0 = 2720t - 16t^2$ a **quadratic equation** in one variable. Each of the following is also a quadratic equation.

$$12y^2 = 6 \qquad 3x^2 - 8x + 9 = 0 \qquad 0 = 2 - r^2$$

As you can see, a variable raised to the second power appears in each of these equations, and this is the highest power of the variable. The standard form of a quadratic equation in one variable is as follows:

$Ax^2 + Bx + C = 0$, where A, B, and C are numbers and $A \neq 0$

Perhaps you recognize that the standard form of a quadratic equation makes use of the standard form of a polynomial. (Refer to page 123 to refresh your memory.) That is, in the standard form of a quadratic equation, one member is a polynomial expressed in standard form; the other member of the equation is 0.

Can $12y^2 = 6$ be expressed as $12y^2 - 6 = 0$? How would you express $2v^2 + 6 = 3v$ in the standard form of a quadratic equation?

In this chapter you will study the systematic solution of quadratic equations (and inequalities). To solve these open sentences, you will make use of your knowledge of real numbers and many ideas you studied in Chapters 3–5.

A Note on Language

In speaking about polynomials and equations involving polynomials, it is frequently convenient to refer to the *degree* of the polynomial. For a polynomial in one variable that is in standard form, the degree of the polynomial is the power of the variable in the *leading term*. Thus, a polynomial such as $12y^2 - 6$ is said to be of the *second* degree, while $3x - 5$ is said to be of the *first* degree. In a similar way, we refer to a quadratic equation such as $12y^2 - 6 = 0$ as a second-degree equation. All quadratic equations are, by definition, second-degree equations. (What degree is an equation such as $3x - 5 = 0$?)

Reviewing Equation-Solving Principles

As preparation for working with quadratic equations, let us review basic principles of equation solving.

1. You know that two equations are equivalent if their solution sets are identical. What is the solution set of $5x = 10$? of $5x + 3 = 13$? Are these equations equivalent?

2. What is the solution set of $3y = 12$? of $6y = 24$? Are these equations equivalent?

3. What is the solution set of $2z = 6$? Is the equation $4z^2 = 36$ satisfied by 3? by -3? Then are the solution sets of $2z = 6$ and $4z^2 = 36$ identical? Are these equivalent equations?

4. From one equation we can obtain equivalent equations by performing certain operations on both members.

a. What operation performed on the members of $5x = 10$ will produce the equivalent equation $5x + 3 = 13$?

b. Is it also possible to operate on the members of $5x + 3 = 13$ to produce $5x = 10$? What operation would we use? If your answer was *subtraction*, would you also agree that this operation can be regarded as addition?

5. What operation performed on the members of $3y = 12$ will produce the equivalent equation $6y = 24$? Can we also operate on the members of $6y = 24$ to produce $3y = 12$? Show that this operation, too, may be regarded as multiplication.

It is possible to operate on the members of an equation to produce a second equation equivalent to it. In such a case, it is always possible to operate on the second equation to produce the first. This is indicated below by the "if and only if" phrase.

▶ The addition principle:

For all values of a, b, and c,
$a = b$ if and only if $a + c = b + c$.

▶ The multiplication principle:

For all values of a, b, and c $(c \neq 0)$,
$a = b$ if and only if $ac = bc$.

6. You have seen that $2z = 6$ and $4z^2 = 36$ are *not* equivalent. The first equation has the solution set $\{3\}$, and the second has the solution set $\{-3, 3\}$.

What operation performed on the members of $2z = 6$ will produce $4z^2 = 36$? Then squaring each member of an equation does *not* necessarily produce an equivalent equation. So we *cannot* say that, for all values of a and b, $a = b$ if and only if $a^2 = b^2$.

In the work that follows, we will study the equation $a^2 = b^2$ and see whether we can find a sentence that is equivalent to it; that is, a sentence for which we can use the "if and only if" phrase.

Investigating $a^2 = b^2$

1. What are the members of the solution set of $x^2 = 25$? You know that 5 is the only positive number whose square is 25. Does it seem reasonable that -5 is the only other member of the solution set? It can be proved that 5 and -5 are, in fact, the only members of this solution set.

2. Consider this disjunction:

$$x = 5 \quad or \quad x = -5$$

Do you remember that a disjunction is true if either (or both) of its sentences is true? Is this disjunction true if x is replaced by 5? by -5? Will it be true if x is replaced by any other number? Then $x^2 = 25$ and the disjunction $x = 5$ *or* $x = -5$ are equivalent sentences.

3. What is the solution set of $y^2 = 16$? What is the solution set of the disjunction:

$$y = 4 \quad or \quad y = -4$$

Are $y^2 = 16$ and the disjunction $y = 4$ *or* $y = -4$ equivalent sentences?

Let us state the following principle.

▶ For all values of a and b,

$a^2 = b^2$ if and only if $a = b$ *or* $a = -b$.

4. Is the quadratic equation $4x^2 = 36$ equivalent to the disjunction $2x = 6$ *or* $2x = -6$? Support your answer by comparing the solution set of $4x^2 = 36$ with the solution set of the disjunction.

EXERCISES

Write disjunctions equivalent to the following equations.

1. $x^2 = 4$	**2.** $x^2 = 1$	**3.** $y^2 = 144$
4. $4x^2 = 49$	**5.** $9z^2 = 4$	**6.** $16x^2 = 25$
7. $25z^2 = 9$	**8.** $100z^2 = 49$	**9.** $z^2 = \frac{1}{4}$
10. $a^2 = \frac{1}{9}$	**11.** $b^2 = .09$	**12.** $c^2 = 1.96$

Write these disjunctions as equivalent quadratic equations.

13. $x = 7$ *or* $x = -7$	**14.** $r = 11$ *or* $r = -11$
15. $2x = 3$ *or* $2x = -3$	**16.** $3a = 2$ *or* $3a = -2$
17. $y = .1$ *or* $y = -.1$	**18.** $5b = 1$ *or* $5b = -1$

Solving Special Quadratic Equations

1. You know that $x^2 = 36$ is equivalent to $x = 6$ *or* $x = -6$. Then what is the solution set of $x^2 = 36$?

2. Is $x^2 = 5$ equivalent to the disjunction below?

$$x = \sqrt{5} \quad or \quad x = -\sqrt{5}$$

Then what is the solution set of $x^2 = 5$?

3. Study the three sentences at the right. Is the disjunction in line (2) equivalent to $4x^2 = 9$? Is the disjunction in line (3) equivalent to the one in line (2)? Why?

(1)	$4x^2 = 9$
(2)	$2x = 3$ *or* $2x = -3$
(3)	$x = \frac{3}{2}$ *or* $x = -\frac{3}{2}$

Then is $\{\frac{3}{2}, -\frac{3}{2}\}$ the solution set of $4x^2 = 9$? Check each of the numbers.

EXERCISES

Find the solution sets of the following quadratic equations. Express irrational roots in radical form. Simplify where possible.

1. $x^2 = 64$	**2.** $y^2 = 60$	**3.** $z^2 = 121$
4. $4x^2 = 81$	**5.** $a^2 = \frac{49}{4}$	**6.** $9b^2 = 64$
7. $x^2 = 24$	**8.** $4x^2 = 35$	**9.** $9x^2 = 45$
10. $36x^2 = 7$	**11.** $3x^2 = 27$	**12.** $3x^2 = 24$
13. $5x^2 = 80$	**14.** $5x^2 = 50$	**15.** $25x^2 = 43$

Sometimes you may be asked for an approximation of an irrational root (rather than the radical form). In Chapter 8 you learned how to determine approximations of square roots. However, for convenience, a table of approximate square roots is generally used. You have probably had experience with such a table in your previous courses in mathematics. In this text, a table of square roots is on page 426. To find the approximate square root of a number such as 37, you merely look in the column headed N until you come to 37; then look across the row to the column headed $\sqrt{N}$. Do you find the entry 6.08? To the nearest tenth, then, what is $\sqrt{37}$? $-\sqrt{37}$?

Solve. Express irrational roots to the nearest tenth.

16. $x^2 = 2$	**17.** $x^2 = 3$	**18.** $x^2 = 5$
19. $5x^2 = 30$	**20.** $6x^2 = 42$	**21.** $2x^2 = 26$
22. $9x^2 = 72$	**23.** $18x^2 = 18$	**24.** $3x^2 = 36$
25. $5x^2 = 55$	**26.** $7x^2 = 70$	**27.** $3x^2 = 0$

Extending the Use of $a^2 = b^2$

In all the equations of the form $a^2 = b^2$ that we have considered thus far, a has always been a monomial such as $2x$ or $3y$. But a binomial such as $x - 3$ can also represent a number. Does it seem reasonable, then, that a may appear as a binomial?

1. If $(x - 3)^2 = 4$, can we think of $(x - 3)^2$ as a^2? What is b^2? Then $(x - 3)^2 = 4$ if and only if:

$$x - 3 = 2 \quad or \quad x - 3 = -2$$

Then is $\{5, 1\}$ the solution set of $(x - 3)^2 = 4$? Check to see that each of these numbers does satisfy this equation.

2. Is $(x - 2)^2 = 5$ equivalent to the disjunction below?

$$x - 2 = \sqrt{5} \quad or \quad x - 2 = -\sqrt{5}$$

Then we have: $x = 2 + \sqrt{5}$ *or* $x = 2 - \sqrt{5}$

What is the solution set of $(x - 2)^2 = 5$?

Let us check to see whether $2 + \sqrt{5}$ does make the sentence $(x - 2)^2 = 5$ true. $(2 + \sqrt{5} - 2)^2$ equals $(\sqrt{5})^2$, or 5. Then the equation is true when x is replaced by $2 + \sqrt{5}$. Check $2 - \sqrt{5}$.

EXERCISES

Solve the following quadratic equations. Express irrational roots in radical form. Check your answers.

1. $(x - 3)^2 = 25$	**2.** $(a + 2)^2 = 49$
3. $(x + 2)^2 = 7$	**4.** $(x - 3)^2 = 11$
5. $8 = (y - 3)^2$	**6.** $15 = (z - 15)^2$

7. If x represents a real number, can $(x - 3)^2$ possibly equal a negative number? Then is $\emptyset$ the solution set of $(x - 3)^2 = -4$?

Solving Quadratic Equations by Factoring

We have been considering the solution of some special quadratic equations. Let us now develop a more general solution of quadratic equations.

1. What is the common name for the product in each of the following?

a. $7 \cdot 3 \cdot 0 \cdot 5$ **b.** $8 \cdot 8 \cdot 2\frac{1}{2} \cdot 0$ **c.** $7.1 \cdot 0 \cdot 6.3$

Did you find that each product was 0? Is it always true that, in a group of factors, if one factor is 0 the product is 0?

2. If three times a certain number is 0, what is the number? If the product of several numbers is 0, what do you know about at least one of the numbers?

Applying the above information to the case of two factors, we can say:

If a and b are real numbers, $ab = 0$ if and only if $a = 0$ or $b = 0$.

Or: A sentence of the form $ab = 0$ is equivalent to the disjunction $a = 0$ *or* $b = 0$.

Can both a and b equal zero?

3. We can use the principle just stated to solve quadratic equations. Suppose, for instance, you are asked to find the solution set of:

$$x^2 - 2x - 8 = 0$$

From your experience in factoring, you know that $x^2 - 2x - 8$ can be expressed as $(x - 4)(x + 2)$. Then the original equation may be written as:

$$(x - 4)(x + 2) = 0$$

As you can see, this equation is of the form $ab = 0$ where $x - 4$ is one factor and $x + 2$ is the other factor. We know now that $(x - 4)(x + 2) = 0$ will be true if and only if:

$$x - 4 = 0 \quad or \quad x + 2 = 0$$

What number will satisfy $x - 4 = 0$? What number will satisfy $x + 2 = 0$? Then the solution set of $x^2 - 2x - 8 = 0$ should be $\{4, -2\}$. Test each of these numbers to see that each does satisfy $x^2 - 2x - 8 = 0$.

4. Look again at the equation $(x - 4)(x + 2) = 0$. Notice that $\{4, -2\}$ is the solution set of this sentence. That is, when x is replaced by 4, we obtain the sentence $(4 - 4)(4 + 2) = 0$; that is, $0 \cdot 6 = 0$. Is this a true statement?

Similarly, when x is replaced by -2, we obtain the sentence $(-2 - 4)(-2 + 2) = 0$; that is, $-6 \cdot 0 = 0$. Is this also a true statement?

5. To solve the quadratic equation $x^2 - 5x = 0$, we proceed as shown at the right. Explain each of the steps. Check to see that the numbers 0 and 5 satisfy the original equation.

$x^2 - 5x = 0$
$x(x - 5) = 0$
$x = 0$ *or* $x - 5 = 0$
Solution set: $\{0, 5\}$

6. Find the solution set of $y^2 + 6y = 0$.

7. To make use of the general principle used in **4–6**, you must have an equation in which the product of two (or more) factors equals 0. Suppose, then, you are asked to solve the equation shown here.

$$x^2 - 2x = 35$$

Your first step is to find an equivalent equation in which one member is 0. Can you obtain such an equation by adding -35 to each member? What equation do you obtain?

Factor $x^2 - 2x - 35$ and complete the solution. Is -5 one member of the solution set of the original equation? What is the other member of the solution set?

8. Can the polynomial $x^2 - 64$ be expressed as $(x + 8)(x - 8)$? Find the solution set of the equation shown at the right.

$$x^2 - 64 = 0$$

9. To solve an equation such as $x(x + 8) = -7$, you should first express it in the form $Ax^2 + Bx + C = 0$. Do this. What equation do you obtain? Complete the solution and check the roots.

Did any of the solution sets in **3–9** have more than two members? A quadratic equation has at most two distinct roots.

10. Solve the equation $x^2 - 4x + 4 = 0$. How many different numbers are roots of this equation? Then the solution set of this equation is written simply as $\{2\}$. However, mathematicians frequently say that such an equation has two equal roots. In this way, statements can be made about the two roots of a quadratic equation.

▶ When a quadratic equation is expressed so that one member is 0 and the solution is found by expressing the other member as the product of two (or more) factors, we call this a solution **by factoring.**

EXERCISES

If you feel that you need to review methods of factoring before proceeding, review pages 136–153.

Find the solution set of each of the following equations, using the factoring procedure. Check each answer in the original equation.

1. $x^2 - 10x + 24 = 0$
2. $y^2 + 11y + 10 = 0$
3. $z^2 + 8z - 9 = 0$
4. $x^2 - 9 = 0$
5. $x^2 - 42 = x$
6. $x^2 - 7x = 0$
7. $25m^2 - 49 = 0$
8. $2y^2 + 3y = 0$
9. $2y^2 + 3y + 1 = 0$
10. $a^2 + 80 = 18a$
11. $x^2 + 9x = 0$
12. $z^2 = 8z - 16$
13. $2x^2 + 23x = 12$
14. $x(x + 5) = 24$

15. Look at the equation $x^2 - 9 = 0$ in Exercise **4**. Suppose, instead of factoring the equation, you had added 9 to each member. What equation would you have obtained? What is the solution set of $x^2 = 9$? Are these results consistent with the results you obtained by factoring? Look again at the equation $x^2 - 64 = 0$ (in **8**, at the top of the page). Solve this by using the principle relating to equations of the form $a^2 = b^2$.

For each of the following, translate the sentence into an equation, solve the equation, and check your answers in the original statement of the problem to be sure they make sense. In certain cases, you will have to discard a root of the equation.

16. The square of x increased by x is 56. What does x equal?

17. The square of y diminished by twice y is 80. What does y equal?

18. The square of a number increased by 9 times that number and then by 25 is 7. What is the number?

19. The product of two consecutive even integers is 20 more than the smaller integer. What are the two integers?

20. The length of a rectangle is 2 inches more than the width. Its area is 63 square inches. What is its width?

Completing the Square

1. Suppose you are asked to solve this equation: $x^2 - 6x = 1$. You might begin by expressing the equation as the equivalent equation $x^2 - 6x - 1 = 0$. Try to factor $x^2 - 6x - 1$. Can it be factored by any method you know?

Then a different method is needed to solve the equation $x^2 - 6x = 1$.

2. Suppose you knew that $x^2 - 6x = 1$ was equivalent to the equation $(x - 3)^2 = 10$. Could you then find the solution as follows? (Look back at your work on pages 294–295.)

$$x - 3 = \sqrt{10} \quad or \quad x - 3 = -\sqrt{10}$$
$$x = 3 + \sqrt{10} \quad or \quad x = 3 - \sqrt{10}$$
$$\text{Solution set: } \{3 + \sqrt{10}, 3 - \sqrt{10}\}$$

3. Let us test the number $3 + \sqrt{10}$ to see whether it satisfies $x^2 - 6x = 1$.

$$(3 + \sqrt{10})^2 - 6(3 + \sqrt{10}) = 1$$
$$(9 + 6\sqrt{10} + 10) - (18 + 6\sqrt{10}) = 1$$
$$1 = 1$$

Test $3 - \sqrt{10}$ in the same way.

4. In your work on pages 294–295, you solved equations such as $(y - 4)^2 = 7$ or $(x + 2)^2 = 9$. When an equation is in this form—that is, when one member of the equation is the square of a binomial and the other member is a number—is the solution easily obtained?

5. Let us look again at this equation. The expression $x^2 - 6x$ is not the square of a binomial. $x^2 - 6x = 1$
But $x^2 - 6x + 9$ is the square of a binomial. It is the square of $(x - \underline{?})$.

We have the equation $x^2 - 6x = 1$. We wish to have the expression $x^2 - 6x + 9$ as the left member of an equation that is equivalent to the original equation. We therefore use the addition principle and add 9 to each member of $x^2 - 6x = 1$. We then have:

$$x^2 - 6x + 9 = 1 + 9$$
$$\text{or} \qquad (x - 3)^2 = 10$$

Is this the equation that was used in **2**?

6. You can see that you have to learn the method for finding the 9 that was added to $x^2 - 6x$. This method is called **completing the square.** Copy and complete the following table on your paper.

(Binomial)²	*Trinomial*	*Coefficient of x*	*The constant term*
$(x - 3)^2$	$x^2 - 6x + 9$	-6	9
$(x + 4)^2$	$x^2 + 8x + 16$	8	?
$(x - 2)^2$	?	?	?
$(x + 1)^2$	?	?	?
$(x - 5)^2$	?	?	?

Do you notice that in each case the coefficient of x^2 in the second column is 1? Try this experiment: Take half of the coefficient of x (the number in the third column) and square it. Is the result the corresponding constant term (the number in the fourth column)?

What do you think should be added to $x^2 + 12x$ to make it a square? This is the square of what binomial? What should be added to $x^2 - 20x$ to make it a perfect square?

Do you agree that to make a perfect trinomial square of $x^2 + Dx$, we add $(\frac{1}{2}D)^2$?

7. To make a trinomial square of $x^2 + \frac{B}{A}x$, would you add $\left(\frac{1}{2} \cdot \frac{B}{A}\right)^2$; that is, $\left(\frac{B}{2A}\right)^2$ or $\frac{B^2}{4A^2}$? Write the trinomial square.

EXERCISES

For each of the following, find the number that can be added to produce a sum that is a trinomial square. Write each trinomial square.

1. $x^2 - 2x$	**2.** $x^2 + 2x$	**3.** $x^2 - 14x$
4. $a^2 + 8a$	**5.** $y^2 - 4y$	**6.** $z^2 + 14z$
7. $x^2 + 3x$	**8.** $y^2 - y$	**9.** $b^2 - 5b$
10. $y^2 + \frac{1}{2}y$	**11.** $z^2 + \frac{2}{3}z$	**12.** $m^2 - \frac{4}{5}m$
13. $x^2 - \frac{1}{2}x$	**14.** $y^2 + \frac{1}{5}y$	**15.** $z^2 - \frac{2}{5}z$

16. When you add 1 to $x^2 - 2x$, the sum is $x^2 - 2x + 1$. This is the square of the binomial $x - 1$. Name the binomial whose square is the trinomial you obtained in Exercise **2**; **3**.

17. Name the binomial whose square is the trinomial you obtained in each of Exercises **4–15**.

Solving Equations by Completing the Square

1. Consider the equation $x^2 - 6x = 11$. What number can be added to the left member to make it a perfect square? If you add 9 to the left member, what do you have to add to the right member to produce an equation that is equivalent to the original equation? What is your new equation? Can you express $x^2 - 6x + 9$ as the square of a binomial? Is the sentence $(x - 3)^2 = 20$ equivalent to this disjunction: $x - 3 = \sqrt{20}$ *or* $x - 3 = -\sqrt{20}$? Can you now decide what numbers are members of the solution set of $x^2 - 6x = 11$?

Your solution to the equation might be written out in a manner similar to that shown here.

(1) $x^2 - 6x = 11$
(2) $x^2 - 6x + 9 = 11 + 9$
(3) $(x - 3)^2 = 20$
(4) $x - 3 = \sqrt{20}$ *or* $x - 3 = -\sqrt{20}$
(5) $x = 3 + \sqrt{20}$ *or* $x = 3 - \sqrt{20}$
(6) Solution set: $\{3 + 2\sqrt{5}, 3 - 2\sqrt{5}\}$

Do you see that each of steps (2)–(5) involves a sentence that is equivalent to each of the others? Show how $\sqrt{20}$ can be expressed as $2\sqrt{5}$.

To check to see whether $3 + 2\sqrt{5}$ is really a root of the original equation, we proceed in a way similar to that used on page 298.

$$(3 + 2\sqrt{5})^2 - 6(3 + 2\sqrt{5}) = 11$$
$$(9 + 12\sqrt{5} + 20) - (18 + 12\sqrt{5}) = 11$$
$$11 = 11$$

Check $3 - 2\sqrt{5}$ to see that it really is a root of the original equation.

A short way of writing the two numbers $3 + 2\sqrt{5}$ and $3 - 2\sqrt{5}$ is $3 \pm 2\sqrt{5}$, which is read "3 plus or minus $2\sqrt{5}$." You may use this notation if you wish.

Solve the following quadratic equations by completing the square. Leave irrational answers in radical form. Check your answers.

	a	b	c
2.	$x^2 + 2x = 35$	$x^2 - 2x = 1$	$z^2 + 8z = 9$
3.	$y^2 + 4y = 5$	$y^2 - 4y = 1$	$z^2 - 10z = 3$

4. To solve the equation $x^2 + 3x - 3 = 0$, we can use the addition principle to write the equivalent equation $x^2 + 3x = 3$. Complete the solution.

5. The equation $x^2 - 3 = -3x$ is equivalent to what equation that you know how to solve? What, then, are the roots of this equation?

Find the solution set of each of the following equations. Leave irrational roots in radical form.

6.	$s^2 + 8s - 1 = 0$	$x^2 - 2x - 2 = 0$	$6m = m^2 + 1$
7.	$y^2 = 10y + 2$	$4z - 2 = z^2$	$14p - 3 = p^2$

8. In completing the square in **2–7**, you may have noticed that the coefficient of the x^2 was always 1. Of course, it is possible to find a perfect trinomial square in which this is not the case. But in general you will find it easier to proceed if the coefficient of x^2 is 1.

By what can you multiply a binomial such as $2x^2 + 4x$ so that the coefficient of x^2 will be 1? Then what equation is equivalent to $2x^2 + 4x = 7$ but has a coefficient of 1 for x^2?

Solve the equation $x^2 + 2x = 3.5$ by completing the square. What are the members of the solution set of $x^2 + 2x = 3.5$? Do you believe that $2x^2 + 4x = 7$ has the same solution set? Check to see whether $-1 + \sqrt{4.5}$ *does* satisfy the equation $2x^2 + 4x = 7$. Check $-1 - \sqrt{4.5}$ also.

EXERCISES

Solve the following. Leave irrational roots in radical form.

1. $m^2 - 10m = 3$	**2.** $r^2 + 14r = -13$	**3.** $x^2 - 3x = 1$
4. $y^2 + 5y = 2$	**5.** $2x^2 = 2x + 15$	**6.** $x^2 = 2x + 4$
7. $2z^2 - 8z = -3$	**8.** $2w^2 + w - 1 = 0$	**9.** $y^2 + 3y = .4$
10. $2x^2 + 5x = 1$	**11.** $3x + 5 = 2x^2$	**12.** $4t^2 = 8t - 1$
13. $10x^2 = 4x + 7$	**14.** $2x^2 = x + .5$	**15.** $x(x - 2) = 5$

The Quadratic Formula

Using the method of completing the square, we can derive a formula for the roots of any quadratic equation. You know that the standard form of a quadratic equation is: $Ax^2 + Bx + C = 0$ where A, B, and C are numbers and $A \neq 0$. Let us find an expression that will show what the roots equal in terms of A, B, and C.

Given: $Ax^2 + Bx + C = 0$

Then an equivalent sentence is: $Ax^2 + Bx = -C$

and $\frac{1}{A}\cdot(Ax^2 + Bx) = \frac{1}{A}\cdot(-C)$

which simplifies to: $x^2 + \frac{B}{A}x = \frac{-C}{A}$

To make the left member a trinomial square, we add $\left(\frac{B}{2A}\right)^2$ to both members.

Then: $x^2 + \frac{B}{A}x + \left(\frac{B}{2A}\right)^2 = \left(\frac{B}{2A}\right)^2 + \frac{-C}{A}$

This equation may be written: $\left(x + \frac{B}{2A}\right)^2 = \frac{B^2}{4A^2} - \frac{C}{A}$

Then: $\left(x + \frac{B}{2A}\right)^2 = \frac{B^2}{4A^2} - \frac{4AC}{4A^2}$

$= \frac{B^2 - 4AC}{4A^2}$

Then: $x + \frac{B}{2A} = \sqrt{\frac{B^2-4AC}{4A^2}}$ *or* $x + \frac{B}{2A} = -\sqrt{\frac{B^2-4AC}{4A^2}}$

$x + \frac{B}{2A} = \frac{\sqrt{B^2-4AC}}{2A}$ *or* $x + \frac{B}{2A} = -\frac{\sqrt{B^2-4AC}}{2A}$

$x = \frac{-B}{2A} + \frac{\sqrt{B^2-4AC}}{2A}$ *or* $x = \frac{-B}{2A} - \frac{\sqrt{B^2-4AC}}{2A}$

$x = \frac{-B + \sqrt{B^2-4AC}}{2A}$ *or* $x = \frac{-B - \sqrt{B^2-4AC}}{2A}$

That is: $x = \frac{-B \pm \sqrt{B^2-4AC}}{2A}$

This formula is called the **quadratic formula.**

1. To use the quadratic formula, the equation must be in the form $Ax^2 + Bx + C = 0$ $(A \neq 0)$.

For instance, the equation $3x^2 + 5x - 2 = 0$ is in this form. What is A in this equation? B? You can see that C is -2, *not* 2.

Now, making the proper replacements in the formula, we have:

$$x = \frac{-5 \pm \sqrt{25 - (4)(3)(-2)}}{2(3)}$$

$$= \frac{-5 \pm \sqrt{49}}{6}$$

That is: $x = \frac{-5 + 7}{6}$ *or* $x = \frac{-5 - 7}{6}$

$x = \frac{1}{3}$ *or* $x = -2$

Solution set: $\{\frac{1}{3}, -2\}$

In this case, we have chosen an equation that could have been solved by factoring. That is, $3x^2 + 5x - 2 = 0$ can be expressed as $(3x - 1)(x + 2) = 0$. Complete the solution. Do you obtain the solution set $\{\frac{1}{3}, -2\}$?

2. To use the quadratic formula in solving $2x^2 - 3x = 4$, you first obtain $2x^2 - 3x - 4 = 0$; that is, the equivalent equation in standard form. What is A? B? C? Complete the solution by using the formula. Follow a procedure similar to that shown in **1** and work carefully to avoid errors.

EXERCISES

Tell what A, B, and C are in each of the following equations.

1. $x^2 + 8x - 3 = 0$	**2.** $x^2 - 8x + 3 = 0$
3. $2x^2 - 6x - 5 = 0$	**4.** $x^2 - x = 0$

5. Solve each of the equations in Exercises **1–4** by using the quadratic formula.

6. Check the solutions of Exercises **1** and **2**, using one of the other procedures you have learned.

Rewrite each of the following equations as an equivalent quadratic equation in standard form. List what A, B, and C are in each case, then solve by using the quadratic formula.

7. $x^2 - 5x = -6$	**8.** $x^2 + 4x = 96$
9. $y^2 - 19 = 2y$	**10.** $80 = 18m - m^2$
11. $(x - 7)(x + 3) = 4$	**12.** $3x^2 - 96 = 4x$
13. $-3x - 9 = -x^2$	**14.** $y(y + 1) - 2y = 4$
15. $2x - \frac{1}{2}x^2 = 0$	**16.** $x^2 - \frac{1}{4}x = \frac{1}{8}$

INEQUALITIES

As you may imagine, a sentence such as $x^2 < 15$ or $y^2 - 36 \geq 0$ is called a **quadratic inequality.** In this part of the chapter, we will investigate the solution of simple quadratic inequalities. Later in the chapter, you will extend your knowledge of quadratic inequalities.

A Principle for $a^2 < b^2$

1. Let us begin with the sentence $x^2 < 9$. Below are listed ten replacements for x. Which of these replacements for x satisfy the sentence $x^2 < 9$, and which do not?

a. -4	**b.** -3	**c.** $-2\frac{1}{2}$	**d.** -1	**e.** 0
f. 1	**g.** 2	**h.** $2\frac{1}{2}$	**i.** 3	**j.** 4

Do only those numbers that are greater than -3 *and* less than 3 satisfy $x^2 < 9$? Do you think that any number that is greater than -3 *and* less than 3 will satisfy the sentence? Another way to say this is: Only those numbers that are *between* -3 and 3 will satisfy the sentence.

We may write: $x^2 < 9$ is equivalent to $-3 < x < 3$. Do you agree that $-3 < x < 3$ is the *conjunction* $-3 < x$ *and* $x < 3$?

2. Consider the sentence $y^2 < 4$. Does -3 satisfy this sentence? -2? 2? 3? Do you think that any number that is *between* -2 and 2 will satisfy $y^2 < 4$? Try a few such numbers—for instance, -1, $-\frac{1}{2}$, $\frac{1}{4}$, 1.9.

We state the following principle.

▶ $a^2 < b^2$ if and only if $-b < a < b$, where b is the positive square root of b^2.

3. Does the solution set of $x^2 < 25$ consist of all numbers between -5 and 5? We may write this solution set as $\{x \mid -5 < x < 5\}$.

What is the solution set of $y^2 < 10$? Represent this solution set in a manner similar to the one shown above.

4. Look at the sentence at the right. Is the number 5 a member of the solution set? the number -5? The solution set may be represented as $\{x \mid -5 \leq x \leq 5\}$.

$x^2 \leq 25$

5. Represent the solution set of $y^2 \leq 10$.

6. Using the general principle stated following **2**, can we say that $4z^2 < 49$ is equivalent to $-7 < 2z < 7$? Is this equivalent to $-\frac{7}{2} < z < \frac{7}{2}$? Then what is the solution set of $4z^2 < 49$?

7. Using the general principle, can we say that $(x + 2)^2 < 9$ is equivalent to $-3 < x + 2 < 3$? Complete the solution. (You may wish to solve $-3 < x + 2$ first, then solve $x + 2 < 3$, and then bring the work together.)

Which of the sets below is the solution set of the original inequality?

a. $\{x \mid -1 < x < 5\}$ **b.** $\{x \mid -5 < x < 1\}$

Check your answer by testing the numbers -5, -3, 0, 1, 2 in the inequality $(x + 2)^2 < 9$.

8. We can draw the graphs of the solution sets of these inequalities on the number line. Below is a graph of $\{x \mid -5 < x < 1\}$. Draw a similar graph of the solution set of $x^2 < 25$; $x^2 \leq 25$.

9. Draw a graph of the solution set of $y^2 \leq 10$. You can use the square root table on page 426 for help in determining where to locate the end points of your graph.

EXERCISES

Solve each of the following and draw the graph of the solution set.

1. $x^2 < 100$	**2.** $4x^2 < 81$	**3.** $9y^2 < 1$
4. $z^2 \leq 8$	**5.** $(x - 1)^2 \leq 25$	**6.** $(x - 2)^2 < 49$
7. $(\frac{1}{2}x + 1)^2 < 1$	**8.** $(\frac{1}{2}x - 1)^2 < 1$	**9.** $(x + 2)^2 < 2$
10. $x^2 - 1 < 36$	**11.** $y^2 - 2 < 4$	**12.** $\frac{3}{4}x^2 \leq 12$

A Principle for $a^2 > b^2$

1. Will the sentence at the right be true if x is replaced by 3? by $3\frac{1}{2}$? by 4? by 5?

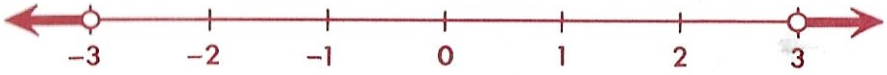
$x^2 > 9$

Will the sentence be true if x is replaced by -3? by $-3\frac{1}{2}$? by -4? by -5?

Will any number that is greater than 3 satisfy $x^2 > 9$? Will any number that is less than -3 also satisfy $x^2 > 9$?

2. Let us form the *disjunction* $x > 3$ *or* $x < -3$. Is this disjunction equivalent to the sentence $x^2 > 9$?

3. Below we have drawn a graph of the solution set of $x^2 > 9$. Why does the graph extend indefinitely to the right? Why does it extend indefinitely to the left?



4. Which of the following are members of the solution set of $y^2 > 4$?

a. -5	**b.** -4	**c.** -3	**d.** -2	**e.** 0
f. 1	**g.** 2	**h.** $2\frac{1}{2}$	**i.** 3	**j.** 4

Is the disjunction $y > 2$ *or* $y < -2$ equivalent to $y^2 > 4$?

Draw a graph of the solution set of $y^2 > 4$.

Do you agree with the following general principle?

▶ $a^2 > b^2$ if and only if $a > b$ *or* $a < -b$, where b is considered to be a positive number.

5. Find the solution of $4x^2 > 25$ and draw the graph of the solution set.

6. Find the solution of $(y + 2)^2 > 36$ and draw the graph of the solution set.

7. Solve each of the following and draw the graph of the solution set. Notice that these sentences involve the symbols $>$, $\geq$, $<$, and $\leq$.

a. $(x - 3)^2 > 16$	**b.** $(x + 4)^2 \geq 49$
c. $(x + 4)^2 \leq 49$	**d.** $(y + 2)^2 < 9$
e. $(x + 3)^2 > 25$	**f.** $(x + \frac{1}{2})^2 > 1$

EXERCISES

Solve each of the following and draw the graph of the solution set.

1. $x^2 > 49$	**2.** $y^2 > 1$	**3.** $a^2 \geq 0$
4. $(x - 1)^2 > 4$	**5.** $(x - 1)^2 \geq 4$	**6.** $(x - 1)^2 \leq 4$
7. $(x - 1)^2 > 3$	**8.** $(2x - 1)^2 > 10$	**9.** $(2x - 1)^2 \leq 10$
10. $x^2 + 1 \geq 26$	**11.** $x^2 + 1 \leq 26$	**12.** $\frac{4}{9}y^2 - 1 \geq 0$

13. To solve an inequality such as $x^2 + 2x + 1 < 5$, you need only recognize that $x^2 + 2x + 1$ is $(x + 1)^2$. Complete the solution. You need not graph the solution set.

14. Find the solution of $x^2 - 8x + 16 > 4$ and draw a graph of the solution set.

Completing the Square: Inequalities

1. Consider the inequality $x^2 - 6x < 7$. If you could obtain an equivalent inequality in which the left member was a perfect square and the right member was some number, could you then find the solution set?

What number can be added to $x^2 - 6x$ to make it a perfect trinomial square? Then what must be added to the right member so that we obtain an inequality equivalent to our original inequality? The trinomial $x^2 - 6x + 9$ is the square of what binomial? Our procedure, including one more step, is shown here. What principle have we used in writing $-4 < x - 3 < 4$? Complete the solution and draw the graph of the solution set.

$$x^2 - 6x < 7$$
$$x^2 - 6x + 9 < 7 + 9$$
$$(x - 3)^2 < 16$$
$$-4 < x - 3 < 4$$

From your graph, tell which of the following numbers are members of the solution set of $x^2 - 6x < 7$.

a. 0	**b.** $-\frac{1}{2}$	**c.** $-1\frac{1}{2}$	**d.** $6\frac{1}{2}$	**e.** 8

Now test the correctness of your answers by using each of these numbers, in turn, as the replacement for x in $x^2 - 6x < 7$.

2. To solve the inequality $x^2 - 6x > 7$, you can again make use of the method of completing the square. Do you obtain $(x - 3)^2 > 16$ as an equivalent inequality?

From your work on pages 305–306, you know that $(x - 3)^2 > 16$ is equivalent to the disjunction $x - 3 > 4$ *or* $x - 3 < -4$. What simpler disjunction is equivalent to this disjunction?

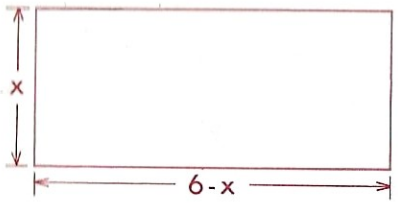
Draw the graph of the solution set of $x^2 - 6x > 7$.

EXERCISES

Solve each of the following inequalities and draw a graph of the solution set. Then see whether the inequality is true when the variable is replaced by a number in the solution set, and false when the variable is replaced by a number that is not in the solution set.

1. $x^2 - 8x < 9$	**2.** $x^2 - 8x > 9$	**3.** $x^2 - 8x \leq 9$
4. $x^2 - 8x \geq 9$	**5.** $x^2 + 10x > -16$	**6.** $x^2 + 10x < -16$
7. $x^2 - 2x \leq 35$	**8.** $x^2 - 2x \geq 35$	**9.** $x^2 - x < \frac{3}{4}$
10. $x^2 - x > \frac{3}{4}$	**11.** $x^2 + 2x < 24$	**12.** $x^2 + 2x > 24$

13. Solve $x^2 - 4x \leq 1$ and graph the solution set. You may use the table on page 426 for help in obtaining an approximation of the numbers corresponding to the end points of the graph.



14. Suppose a set of rectangles consists of all the rectangles having a perimeter of 12 inches. If x represents the length of one side of a rectangle, does $6 - x$ represent the length of the adjacent side? What polynomial represents the area of each rectangle? Is one figure in this set of rectangles a square? What is the length of its side? What is its area?

Suppose we want to find which, if any, of the rectangles in the set has an area greater than that of the square. We begin by writing the condition $x(6 - x) > 9$ and then solve to find the values of x that satisfy the condition.

Is $x(6 - x) > 9$ equivalent to each of the sentences below? Tell why.

(1) $6x - x^2 > 9$
(2) $0 > x^2 - 6x + 9$
(3) $0 > (x - 3)^2$

Examine sentence (3). Can the square of any real number be less than 0? Then have we proved that there is no value of x for which $x(6 - x) > 9$ is a true statement? Then is there any rectangle that has an area greater than that of the square?

15. Prove that no rectangle with a perimeter of 24 inches has an area greater than the area of the square that is a member of this set of rectangles.

GRAPHS OF QUADRATIC FUNCTIONS

The Rule $y = Ax^2 + Bx + C$

You will recall (pages 215–216) that a function is a set of ordered pairs of numbers such that the first number in each pair is associated with only one second number. You saw that if m and b represent numbers, $y = mx + b$ represents a set of equations for linear functions whose graphs are straight lines. Note that $mx + b$ is a polynomial of the first degree.

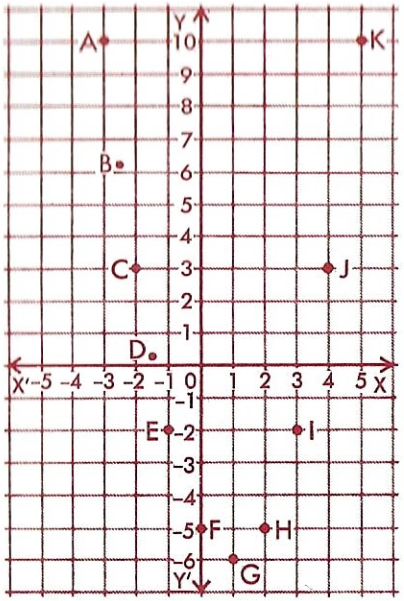
1. To gain a better understanding of quadratic equations and inequalities, we shall study **quadratic functions.** The rule for such a function involves a quadratic polynomial, as in the sentence $y = x^2 - 2x - 5$. The function is the solution set of the sentence; that is, it is the set of ordered pairs (x,y) that satisfy $y = x^2 - 2x - 5$.

To find an ordered pair (x,y) that is a member of the solution set, we choose a replacement for x and find what y must equal. Suppose -3 is chosen as a replacement for x. Then $y = (-3)^2 - 2(-3) - 5$, and y must equal $\underline{?}$. Thus $(-3,10)$ is a member of the solution set.

What ordered pair do we obtain when x is replaced by -2? by 0? In answering such questions, your work can be arranged systematically as follows:

x-replacement		*value of y*	(x,y)
-3	$y = (-3)^2 - 2(-3) - 5$	10	$(-3,10)$
$-2\frac{1}{2}$	$y = \left(\frac{-5}{2}\right)^2 - 2\left(-\frac{5}{2}\right) - 5$	$6\frac{1}{4}$	$\left(-2\frac{1}{2},6\frac{1}{4}\right)$
-2	$y = (-2)^2 - 2(-2) - 5$	3	$(-2,3)$
$-1\frac{1}{2}$	$y = \left(\frac{-3}{2}\right)^2 - 2\left(-\frac{3}{2}\right) - 5$	$\frac{1}{4}$	$\left(-1\frac{1}{2},\frac{1}{4}\right)$
-1	$y = (-1)^2 - 2(-1) - 5$	-2	$(-1,-2)$
0	$y = (0)^2 - 2(0) - 5$	-5	$(0,-5)$
1	$y = (1)^2 - 2(1) - 5$	-6	$(1,-6)$
2	$y = (2)^2 - 2(2) - 5$	-5	$(2,-5)$
3	$y = (3)^2 - 2(3) - 5$	?	?
4	$y = (4)^2 - 2(4) - 5$	?	?
5	$y = (5)^2 - 2(5) - 5$	?	?

Do you see that when you are given the first number of an ordered pair (x,y) you can always find exactly one second member such that the ordered pair is a member of the solution set of $y = x^2 - 2x - 5$? Does this mean that the set is a function? Find the remainder of the number pairs necessary for the table.



To draw a graph of the function on page 309, we begin by plotting the points corresponding to the number pairs we have obtained. These points have been plotted in the figure at the left.

There are some questions we must consider before we can draw the graph.

2. Is point G the lowest point? Since the coordinates of G are $(1,-6)$, we might try to answer this question by finding the y-coordinate when $x = .9$ and when $x = 1.1$. Let us try .9 and see whether the y-value we obtain is less than -6.

Replacing x by .9, we have: $y = (.9)^2 - 2(.9) - 5$.

Is the y-value less than -6?

Try 1.1 as a replacement for x. Is the y-value less than -6? Does it seem that G is the lowest point?

3. We can prove that G is the lowest point as follows: Consider $x^2 - 2x - 5$, the right member of the equation. Can we express $x^2 - 2x - 5$ as $x^2 - 2x + 1 - 6$? But $x^2 - 2x + 1 - 6$ is simply $(x - 1)^2 - 6$.

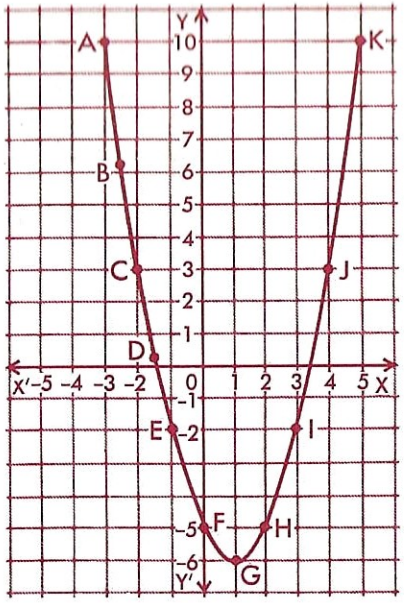
Can the square of a real number ever be negative? What is the smallest number that the square of a real number can be? Then is 0 the smallest number that $(x - 1)^2$ can be? Is -6, then, the smallest number that $(x - 1)^2 - 6$ can be?

For what value of x will $(x - 1)^2$ equal 0? Then the point $(1,-6)$ is the lowest point of the graph.

4. How shall we draw the graph? The points we have plotted suggest a curve. You probably suspected the graph would not be a straight line, since the rule is *not* of the form $y = mx + b$. Since the rule is not that of a linear function, we should not draw a straight line segment between any two of the points such as H and I. Find the y-value when x is replaced by $2\frac{1}{2}$. Is $(2\frac{1}{2},-3\frac{3}{4})$ a point on the graph? This point is between $(2,-5)$ and $(3,-2)$.

Find the point that has an x-coordinate of $3\frac{1}{2}$; an x-coordinate of $4\frac{1}{2}$. Do you think you have enough points to see the shape of the curve?

This curve, shown at the right, is called a **parabola.** We have drawn this parabola only for values of x from -3 to 5, inclusive. It can, of course, be extended.



5. Consider the function whose rule is $y = x^2 - 3$. Is $x^2 - 3$ a quadratic polynomial? Then the rule $y = x^2 - 3$ is the rule for a quadratic function. Find the value of y for the x-replacements indicated in this table.

x	-4	-3	-2	-1	0	1	2	3	4
y									

Draw the graph.

Why must $(0,-3)$ be the lowest point on the graph?

6. Let us investigate the function whose rule is $y = -x^2$. When x is replaced by -3, x^2 is 9, and $-x^2$ is -9. Thus, y equals -9, and $(-3,-9)$ is one member of the function. When x is replaced by -2.5, does y equal 6.25, or does y equal -6.25? Then $(-2.5, -6.25)$ is another member of the function.

Make a table and find the ordered pairs when x is replaced by -2, -1.5, -1, 0, 1, 1.5, 2, 2.5, and 3.

Plot the points and draw the graph of $y = -x^2$. Does the graph have a highest point rather than a lowest point? What is this point?

EXERCISES

Make a table of values and draw the graph of each function whose rule is given. Use values of x as indicated in parentheses.

1. $y = x^2 + 2x - 4 \quad (-4 \leq x \leq 2)$
2. $y = x^2 - 4x \quad (-1 \leq x \leq 5)$
3. $y = x^2 + 2x \quad (-3 \leq x \leq 1)$
4. $y = x^2 + 4x \quad (-5 \leq x \leq 1)$
5. $y = x^2 + 2x - 1 \quad (-3 \leq x \leq 1)$
6. $y = 6 - x^2 \quad (-3 \leq x \leq 3)$
7. $y = x^2 + 4x + 1 \quad (-4 \leq x \leq 0)$
8. $y = x^2 - 6 \quad (-3 \leq x \leq 3)$

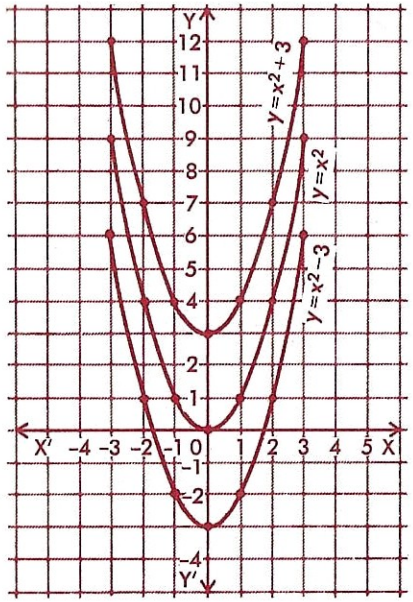
Lowest Points and Highest Points

1. Some parabolas have a lowest point; some have a highest point. Have you noticed a pattern in the equation of a parabola that helps tell whether it has a highest or a lowest point? Draw the graph of each function whose rule is given below.

a. $y = x^2 + 1$ **b.** $y = -x^2 + 1$
c. $y = 2x^2$ **d.** $y = -2x^2$
e. $y = 3x^2 - 4x$ **f.** $y = -3x^2 - 4x$

Which of these have lowest points? Which have highest points?

The graph of a quadratic function with a rule of the form $y = Ax^2 + Bx + C$ $(A \neq 0)$ has a lowest point if the coefficient of x^2 is positive. The graph has a highest point if the coefficient of x^2 is negative.



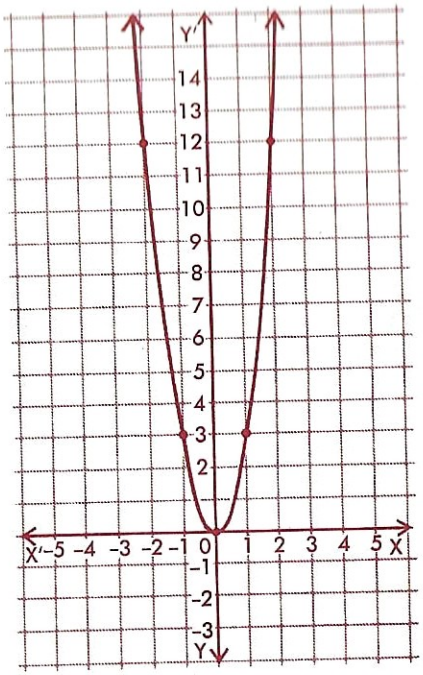
2. Two parabolas may differ only in their positions. At the left are the graphs of the functions whose rules are $y = x^2$, $y = x^2 + 3$, and $y = x^2 - 3$. Compare the y-coordinates of each graph when x is replaced by 1; by -1; by 0. Do you see why we may think of $y = x^2 + 3$ as $y = x^2$ moved up 3 spaces? How could you think of $y = x^2 - 3$ as compared with $y = x^2$?

3. Make a graph of the function whose rule is $y = x^2 + x$. With the help of this graph, sketch the graph of the function whose rule is $y = x^2 + x + 3$; $y = x^2 + x - 3$.

4. Make a graph of the function whose rule is $y = 2x^2 - 2x$. Use this graph to sketch the graph of the function whose rule is $y = 2x^2 - 2x - 2$; $y = 2x^2 - 2x + 2$.

5. Make a graph of the function whose rule is $y = 3x - x^2$. Use this graph to sketch the graph of the function whose rule is $y = 2 + 3x - x^2$; $y = 4 + 3x - x^2$.

Domain and Range



1. Shown here is the graph of the function whose rule is $y = 3x^2$ If we replaced x by any negative number, such as -4, would we obtain a positive number as the value of y? Would we also obtain a positive value for y if we replaced x by a positive number such as 12? Neither $(-4,48)$ nor $(12, 432)$ is shown on the graph, but the arrowheads indicate that the graph extends indefinitely as indicated. What is the y-coordinate of the lowest point of the graph? Can we ever obtain a number less than 0 as the value for y?

From this discussion and from the graph, do you see that, if we do not limit the replacement for x, the domain of the function $\{(x,y) \mid y = 3x^2\}$ is the set of real numbers, but the range is the set of non-negative real numbers?

Let us agree that in the following work the domain of x will be the set of real numbers.

2. a. Consider the function whose rule is $y = 3x^2 - 1$. Draw the graph. What is the range of this function?
b. What is the range of the function whose rule is $y = 3x^2 + 2$?

3. a. Draw the graph of the function whose rule is $y = -3x^2$. What is the range of this function?
b. What is the range of the function whose rule is $y = -3x^2 + 2$?

Look over your work on pages 309–312. Do you see that in considering a function whose rule is $y = Ax^2 + Bx + C$ $(A \neq 0)$ it is useful to know the lowest (or highest) point on the graph? Reread the discussions in **3–6** on pages 310–311 to help your understanding of *range*.

Using Graphs to Solve Quadratic Equations

1. You have seen how a number pair of a quadratic function can be found for any x-replacement. How can we find a number pair for a given y-replacement? For instance, suppose the rule for a quadratic function is $y = x^2 + 2x - 2$ and we wish to find the x-replacement when y equals 6.

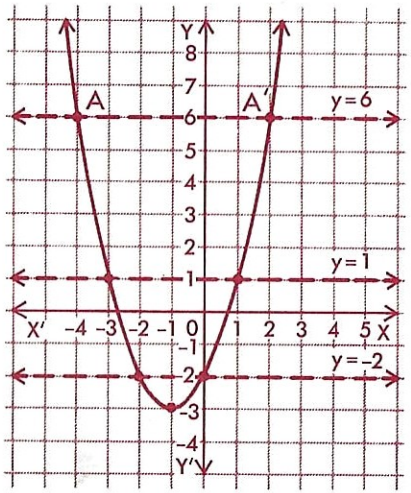
Replacing y by 6, we have: $6 = x^2 + 2x - 2$
This is equivalent to: $0 = x^2 + 2x - 8$

Can you find the solution by factoring? Are there two x-replacements that satisfy $6 = x^2 + 2x - 2$? Name these replacements.

Suppose you wished to find the number pair (or pairs) when y equals 4. You might then write:

$$4 = x^2 + 2x - 2$$
$$0 = x^2 + 2x - 6$$

Can you solve this equation by factoring? If not, use the quadratic formula. What results do you obtain?



2. An engineer working with $y = x^2 + 2x - 2$ might find it necessary to replace y by many numbers and find the solution to all of these quadratic equations. If the graph of $y = x^2 + 2x - 2$ is drawn, the corresponding x-replacements can be found quite easily.

At the left is the graph of $y = x^2 + 2x - 2$. All points for which the y-coordinates are 6 are on the dashed line $y = 6$. The points found on this line and on the graph of $y = x^2 + 2x - 2$ are labeled A and A'. The coordinates of A are $(-4,6)$. What are those of A'? Are there any other points on the graph for which the y-coordinate is 6? The only corresponding x-coordinates, then, are -4 and 2.

3. Using the graph in **2**, find x in $y = x^2 + 2x - 2$ when y is replaced by 1; by -2; by -3.

4. Using the graph in **2**, find an approximation of x to the nearest $\frac{1}{4}$ when y is replaced by 3; by 0; by -1.

5. Suppose you were asked to find the x-replacement (or replacements) when y is replaced by -4. You would then be looking for a number that satisfied the sentence $x^2 + 2x - 2 = -4$. Imagine the line that is the graph of $y = -4$. Does this line have any points in common with the graph of $y = x^2 + 2x - 2$? Then there is no real number that satisfies $x^2 + 2x - 2 = -4$.

EXERCISES

1. Suppose you wish to solve $x^2 + 2x - 1 = 0$. You can find the solution set of $x^2 + 2x - 1 = 0$ by first making the graph of $y = x^2 + 2x - 1$. Draw this graph for $-4 \leq x \leq 2$. The problem then is to find the x-coordinates for $y = 0$ on this graph. Find x to the nearest $\frac{1}{4}$.

2. You can also solve $x^2 + 2x - 1 = 1$ by using the graph you made for Exercise **1**. From the graph, find what x must be if $y = 1$. (Approximate to the nearest $\frac{1}{4}$.)

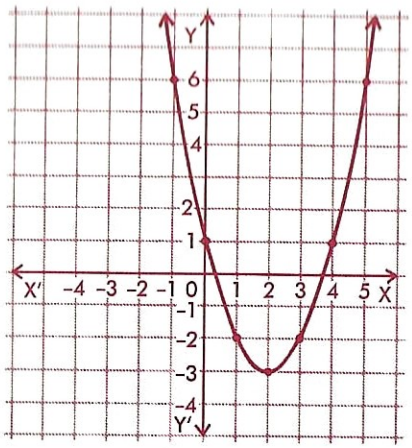
3. Using the graph for Exercise **1**, solve:
   a. $x^2 + 2x - 1 = -1$ b. $x^2 + 2x - 1 = 2$

4. Using the graph for Exercise **1**, solve for x to the nearest $\frac{1}{4}$.
   a. $x^2 + 2x - 1 = 3$ b. $x^2 + 2x - 1 = 5$

5. Make a table of number pairs for $y = x^2 - 3x - 4$. Include $1\frac{1}{2}$ as an x-replacement. It will help to locate the lowest point of the graph. Make this graph and solve each of the following. Approximate to the nearest $\frac{1}{4}$ if necessary.
   a. $x^2 - 3x - 4 = 0$ b. $x^2 - 3x - 4 = -4$
   c. $x^2 - 3x - 4 = 4$ d. $x^2 - 3x - 4 = 5$

6. Make a table of number pairs for $y = x^2 + 6x + 2$. Include -3 as a replacement for x. It will help to locate the lowest point of the graph. Make this graph and solve each of the following. Approximate to the nearest $\frac{1}{4}$ if necessary.
   a. $x^2 + 6x + 2 = 0$ b. $x^2 + 6x + 2 = -5$
   c. $x^2 + 6x + 2 = 3$ d. $x^2 + 6x + 2 = -1$

7. What is the solution set of $x^2 + 6x + 2 = -9$?



Using Graphs to Solve Inequalities

1. We can also use graphing to solve an inequality such as $x^2 - 4x + 1 < 0$. We begin by drawing the graph of $y = x^2 - 4x + 1$. This graph is shown at the left. To solve the inequality $x^2 - 4x + 1 < 0$, we look for the x-replacements that correspond to negative values of y; that is, we look for the x-coordinates of the points that are below the x-axis.

Does y equal 0 when x is approximately .3? Does y equal 0 when x is approximately 3.7? Is y negative when x is greater than .3 *and* less than 3.7? How else might you describe the solution set of $x^2 - 4x + 1 < 0$?

2. Suppose you are asked to solve the inequality $x^2 - 4x + 1 > 0$. Now you are looking for the x-replacements that correspond to positive values of y. Is y positive when x is less than .3? when x is greater than 3.7?

Then the solution set of $x^2 - 4x + 1 > 0$ may be described as the disjunction $\{x \mid x < .3 \text{ or } x > 3.7\}$.

3. To check the solution of $x^2 - 4x + 1 < 0$ (or $x^2 - 4x + 1 > 0$), you can use the method of completing the square (discussed on page 307). Is this solution consistent with the results obtained in 1?

$$x^2 - 4x + 1 < 0$$
$$x^2 - 4x < -1$$
$$x^2 - 4x + 4 < -1 + 4$$
$$(x - 2)^2 < 3$$
$$-\sqrt{3} < x - 2 < \sqrt{3}$$
$$2 - \sqrt{3} < x < 2 + \sqrt{3}$$

EXERCISES

1. Using the graph of $y = x^2 - 4x + 1$, state the solution set of each of the following.

a. $x^2 - 4x + 1 < 2$ **b.** $x^2 - 4x + 1 > 2$

2. Draw the graph of $y = x^2 - x + 4$ and use the graph to find the solution set of each of the following.

a. $x^2 - x + 4 \leq 0$ **b.** $x^2 - x + 4 \geq 0$
c. $x^2 - x + 4 < 3$ **d.** $x^2 - x + 4 > 3$

PROBLEM SOLVING

In the remaining pages of the chapter, you will find problems of various kinds. You will have to decide which method of solution is most convenient, and you will also have to interpret whether the number or numbers you obtain in each case are "answers" for the problem situation.

1. If 3 is subtracted from 4 times a certain number, the result is the square of the number. Find the number or numbers that satisfy this condition.

2. There are two consecutive negative integers whose product is 462. Find these integers.

3. There are three consecutive integers such that the square of the first increased by the product of the other two gives 154. What are they?

4. In the set of positive integers, there are two consecutive even integers such that the square of the second decreased by twice the first leaves 172. What are these integers?

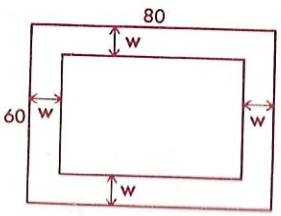
5. The sum of a certain number and its reciprocal is $\frac{25}{12}$. Find the number or numbers.

6. The hypotenuse of a right triangle is 25 feet long. The difference between the lengths of the other two sides is 5 feet. Find the lengths of the sides of this triangle.

7. Show that the difference between the squares of two consecutive integers is an odd integer. If x^2 represents the square of the smaller integer, then $(x + 1)^2$ represents the square of the other integer.

8. A lawn is 60 feet by 80 feet. A uniform strip is cut around it, leaving a rectangle that has half the area of the lawn. How wide is the strip? If w represents the width of this strip, do the expressions $60 - 2w$ and $80 - 2w$ represent the dimensions of the rectangular area that is still uncut?

What is the area of this rectangle? Form the equation and solve.



9. How wide a uniform border should be plowed around a rectangular field 20 rods by 40 rods to leave one-quarter of the field still unplowed?

10. A boy walks a distance of 4 miles at a certain average speed. He returns to his starting point at a speed that is one mile per hour faster. The total time is 1 hour 48 minutes. If x represents the first speed in miles per hour, what expression represents the second speed?

Does $\frac{4}{x}$ represent the time (in hours) it took to travel the 4 miles at his original speed? Does $\frac{4}{x+1}$ represent the time (in hours) it took him to return? What represents the total time in hours? Write the equation, solve, and check.

11. A boy can row downstream 3 miles per hour faster than he can row upstream. He finds that he can row 1 mile upstream and back to his starting point in one hour. What is his rate downstream?

12. A motorboat traveling at 9 miles per hour in still water goes 12 miles downstream and comes back in a total of 3 hours. What is the rate of the current?

13. When Dinah does a certain piece of work, she takes 3 hours longer than Janet does. Together they can do it in 2 hours.

Let x represent the number of hours it takes Janet alone to do the job. What represents the number of hours it takes Dinah alone? What represents the part of the job Janet can do in 1 hour? in 2 hours?

What expression represents the part of the job Dinah can do in 1 hour? in 2 hours? Write the equation that says the job is done in 2 hours if they work together. Solve and check.

14. Marion takes 5 days more to do a job than Joseph. Together they can do it in 6 days. How long does it take each alone to do the job?

15. The formula $h = 2720t - 16t^2$ expresses the height h (in feet) of a projectile after t seconds if the projectile was fired directly upward at a speed of 2720 feet per second. After how many seconds will the object reach a height of 9324 feet?

Find the solution set of each of the following. Use the table on page 426 to approximate to the nearest tenth when necessary. 292–297

Extra Practice Page 407

	a	b	c
1.	$x^2 = 169$	$5y^2 = 80$	$5z^2 = 50$
2.	$4x^2 = 81$	$7x^2 - 343 = 0$	$(x-4)^2 = 25$
3.	$x^2 - 2x - 35 = 0$	$x^2 - 7x + 12 = 0$	$x^2 - 4x - 12 = 0$

Challenge Page 422

Solve each of the following. You may leave irrational answers in radical form, but simplify. 298–303

4.	$x^2 - 6x = 16$	$x^2 - 6x = 1$	$y^2 + 4y = 4$
5.	$y^2 = 2y + 3$	$x^2 - x - 1 = 0$	$2x^2 - 5x + 2 = 0$
6.	$3x^2 - 7x + 2 = 0$	$x + \frac{1}{x} - 4 = 0$	$\frac{x+3}{x-6} = \frac{x}{x-3}$

Solve each of the following inequalities. Draw the graph of the solution set. 304–308

7.	$x^2 > 25$	$x^2 < 25$	$x^2 - 7x + 10 > 0$
8.	$y^2 - 2y \geq 0$	$x^2 + 8x > 20$	$y^2 - 6y < 2$
9.	$(x-3)^2 < 81$	$(2y+1)^2 \leq 169$	$(3r-2)^2 \geq 196$

10. Draw the graph of the function whose rule is $y = x^2 - 4x - 5$. The domain of the function is the set of real numbers. What is the range of the function? 309–313

11. Using the graph you drew in **10**, solve the following. You may approximate to the nearest $\frac{1}{4}$ when necessary. 314–316

a. $0 = x^2 - 4x - 5$ **b.** $-8 = x^2 - 4x - 5$

c. $x^2 - 4x - 5 < 0$ **d.** $x^2 - 4x - 5 > -8$

12. The difference between a positive number and its reciprocal is $\frac{16}{15}$. What is the number? 317–318

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

10 Geometry

In Chapter 6 we used points of a coordinate plane to picture ordered pairs of numbers. This enabled us to plot the graphs of relations and solution sets of open sentences. These graphs, which are lines, points, or regions of the plane, give us a better understanding of relations and open sentences.

In this chapter we will again make use of the one-to-one correspondence between a set of ordered pairs of numbers and a set of points in a coordinate plane. But this time we will use the number pairs to help us investigate some properties of geometric figures.

COORDINATE GEOMETRY

Geometry on a Line

In order to arrive at some basic ideas, let us first consider a single line. You have already seen how a number line is constructed and you know that the points on such a line have a one-to-one correspondence with the set of real numbers. This one-to-one correspondence serves as a basis for a *coordinate system on a line*. In the diagram, point P is located at 2. The number 2 is called the *coordinate* of P. The number 5 is the coordinate of Q. What is the coordinate of W? of R?

Line Segments

You know that a line segment is a part of a line and consists of two distinct points (*end points*) and all the points lying between these points. You know also that we speak of the *length* of a line segment; that is, the distance from one end point to the other.

1. On the number line on page 320, is Q three units distant from P? Is there another point three units distant from P? What is its coordinate?

Let us consider RP and PQ as line segments. What is the distance from R to P? the distance from P to Q? What relationship exists between the measures of line segments RP and PQ?

Since it is awkward to keep saying "the measure of the line segment RP," we will use the notation mRP to mean "the measure of the line segment RP." Thus, mRP is read "the measure of RP." Because a line extends indefinitely, we do not speak of the measure of a line. Thus, if we discuss mRP, we mean a line segment RP.

2. In **1**, you started with point P and found the points, Q and R, three units distant from P. Assume now that P at 2, Q at 5, R at -1, and W at 4 are given. What is the distance from R to Q? from P to W? from R to W? from W to Q?

3. How did you find the distances? Did you find the distance from P to W by subtracting 2 from 4? the distance from R to W by subtracting -1 from 4? Do you think a rule for determining the distance between two points on a line is: "Subtract the coordinate of the point named first from the coordinate of the point named last"? However, suppose you are asked the distance from W to P; by following the rule above, you get $2 - 4$, or -2. And for the distance from W to R you would obtain: $-1 - 4 = \underline{?}$. But mPW should equal mWP, and mRW should equal mWR. Also, we do not wish to have negative numbers as distances. It appears, therefore, that our rule is not satisfactory.

4. Suppose we revise our rule to take advantage of absolute value. Does $|2 - 4| = |4 - 2|$? Does $|-1 - 4| = |4 - (-1)|$? Count the intervals between P and W. Does $mPW = mWP$? What is mRW? mWR?

5. Consider two points: A with coordinate a and B with coordinate b. What two expressions describe the distance between A and B? Is it true that $|a - b| = |b - a|$? Then let us agree: To find the distance between two points, we will find the absolute value of the difference of their coordinates.

▶ In a coordinate system on a line, the distance between point A with coordinate a and B with coordinate b is given by the expression $|a - b|$.

EXERCISES

1. Given point A at $1\frac{1}{2}$, B at -3, C at $6\frac{1}{2}$, D at -8, and E at 11, find the distance:

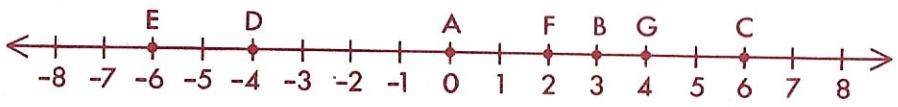
a. from A to C	**b.** from E to A
c. from C to D	**d.** between D and E
e. between D and A	**f.** from B to C
g. between C and B	**h.** from C to E

Find the lengths of line segments with the following end points.

	a	b	c
2.	6 and 10	0 and 4	-2 and -6
3.	-2 and 4	$3\frac{1}{4}$ and $6\frac{1}{2}$	-8.2 and $-.3$
4.	1005 and -1	-38 and 38	$\frac{3}{4}$ and $-\frac{1}{2}$

Line Segments and Midpoints

1. Consider the number line below. Does $mDA = mAG$? We say that A is the **midpoint** of line segment DG. The point A **bisects** DG.



What is the midpoint of line segment AC? of line segment FC?

2. Let a represent the coordinate of one of the end points of a line segment; let b represent the coordinate of the other end point. Evaluate the expression $\frac{a+b}{2}$ for the segment DG; segment AC; segment FC. Do you find that in each case $\frac{a+b}{2}$ is the coordinate of the midpoint of the segment? Choose a few other examples and test them.

▶ If the coordinate of A is a and that of B is b, the coordinate of the midpoint of segment AB is $\frac{a+b}{2}$.

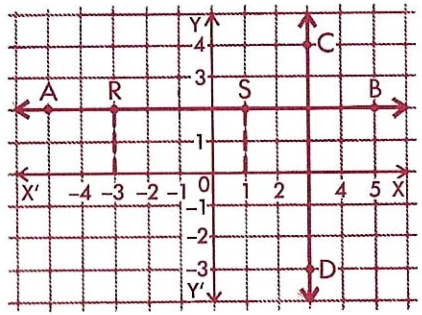
Find the midpoint of line segment PQ if the respective coordinates of P and Q are as follows:

	a	b	c
3.	3 and 10	-3 and 10	-3 and -10
4.	3 and -10	5 and -2	-100 and 100
5.	$5\frac{1}{2}$ and $-1\frac{1}{2}$	$3\frac{1}{2}$ and 3	3 and $-3\frac{1}{2}$

Horizontal and Vertical Lines in the Plane

Let us turn now to geometry in the coordinate plane.

1. You know that a line parallel to the x-axis contains only points whose y-coordinates are the same. You also know that a line parallel to the y-axis contains only points whose x-coordinates are the same number. In the illustration at the right, we have:



line AB, the graph of $\{(x,y) \mid y = 2\}$

line CD, the graph of $\{(x,y) \mid x = 3\}$

What are the coordinates of the point of intersection of lines AB and CD?

2. Consider $R(-3,2)$ and $S(1,2)$. Are R and S on AB? What is the distance between R and S? Does $mRS = |-3-1|$? $|1-(-3)|$?

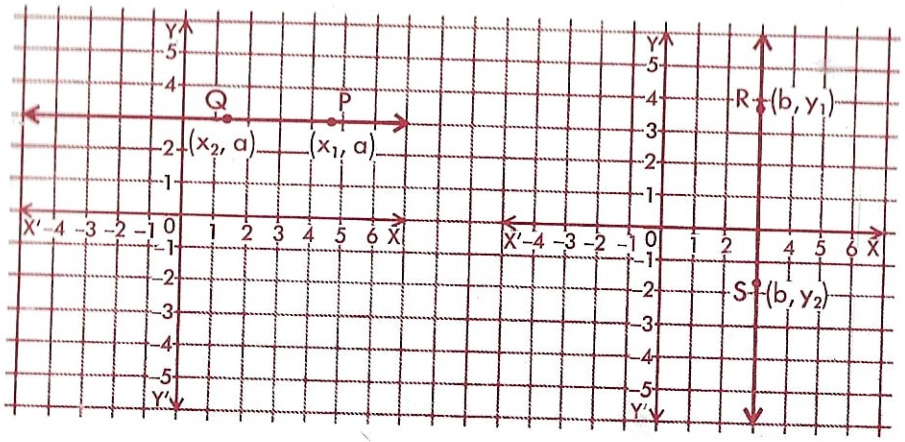
3. What is the distance between points $(3,0)$ and $(7\frac{1}{2},0)$? between $(2,0)$ and $(-3,0)$? between $(-1,0)$ and $(-4,0)$?

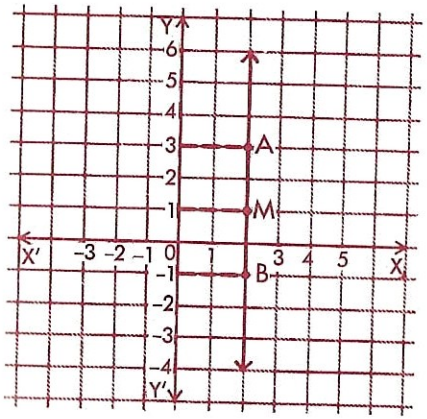
4. What is the distance between points $(0,1)$ and $(0,5)$? between $(0,2)$ and $(0,-2)$? between $(0,-1)$ and $(0,-3\frac{1}{2})$?

5. What is the distance between $(2,1)$ and $(5,1)$? Sketch the position of these two points. Do they both have the same y-coordinate? Do they lie on a line parallel to the x-axis?

6. What is the distance between $(-2,3)$ and $(3,3)$? What is the distance between $(2,3)$ and $(2,7)$? between $(8,2)$ and $(8,-10)$?

7. If two points are given as (x_1,a) and (x_2,a), can you say that the distance between the points is $|x_1 - x_2|$? What is the distance between (b,y_1) and (b,y_2)?





8. In the diagram at the left, line AB is parallel to the y-axis. What are the coordinates of point A? of point B? What are the coordinates of the midpoint of segment AB?

Is the y-coordinate of the midpoint equal to $\frac{3+(-1)}{2}$; that is, 1?

9. Suppose you are asked to find the coordinates of the midpoint of the line segment whose end points are (5,4) and (5,8). You know immediately that the line segment is on a line parallel to the y-axis, and that the x-coordinate of the midpoint is 5. What is the y-coordinate? Is it $\frac{8+4}{2}$; that is, 6?

For a line segment with end points (b,y_1) and (b,y_2), are the coordinates of the midpoint $(b, \frac{y_1+y_2}{2})$?

10. Now suppose you wish to find the midpoint of a line segment with end points (−4,2) and (6,2). You know the y-coordinate of the midpoint is 2. Is the x-coordinate 5, or is it 1?

Plot the graph of $\{(x,y) \mid y = 2\}$. Mark the points (−4,2) and (6,2) on your graph. Mark the midpoint of the segment.

Write an expression that gives the coordinates of the midpoint of a line segment whose end points are (x_1,a) and (x_2,a).

EXERCISES

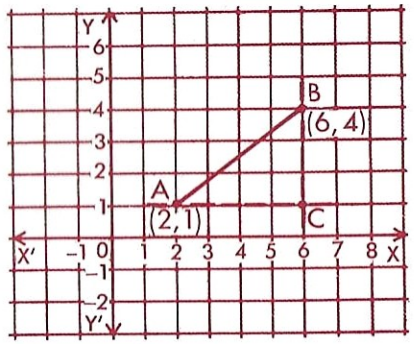
Tell whether each of the following pairs of points is on a line parallel to the x-axis, or to the y-axis. Then give the distance between each pair and the midpoint of the line segment they determine.

1. (3,2) and $(7\frac{1}{2},2)$
2. (5,6) and (5,−1)
3. (−2,3) and (3,3)
4. (8,2) and (8,−10)
5. $(12,\frac{1}{2})$ and $(12,-\frac{1}{2})$
6. $(\frac{3}{4},8)$ and $(\frac{3}{4},4\frac{1}{2})$
7. (6.5, 3.2) and (−6.4, 3.2)
8. $(3\frac{1}{2},-2)$ and $(-3\frac{1}{2},-2)$

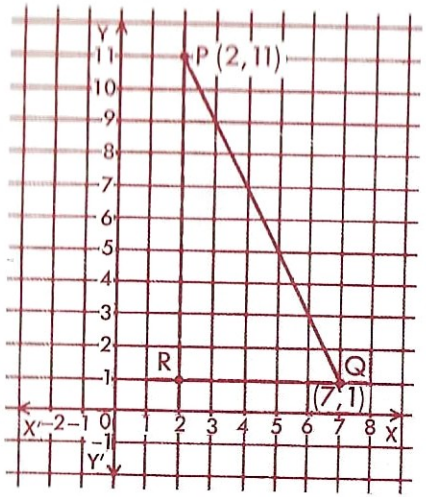
Distance in the Plane

You know that a point in the coordinate plane can be represented by a number pair. You know also that two points determine a line segment, but you have no real method for studying that segment.

1. Suppose the two points under discussion are $A(2,1)$ and $B(6,4)$.



Through A we draw a line parallel to the x-axis; through B, a line parallel to the y-axis. Call the intersection of the two lines point C. Triangle ABC is a right triangle. Since lines AC and BC are parallel to the x- and y-axes, respectively, will the x-coordinate of C be the same as the x-coordinate of B? Will the y-coordinate of C be the same as the y-coordinate of A? What are the coordinates of the vertex at C? Then what is mAC? mBC? You know that the Pythagorean rule tells us that $mAB^2 = mAC^2 + mBC^2$. Make the correct replacements for mAC^2 and mBC^2 and solve for mAB. Is AB 5 units in length?

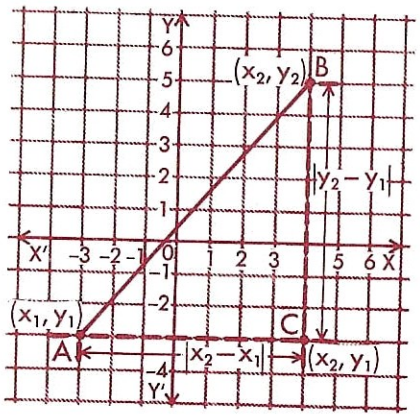


2. Try another example. Take point P at (2,11) and point Q at (7,1). Construct right triangle PRQ. Are the coordinates of R (2,1)? Does $mRQ = 5$? What is mRP? Then do you agree that $mPQ = \sqrt{5^2 + 10^2}$? What is mPQ (in radical form)?

3. Find the distance between points $A(4,1)$ and $B(5,2)$; between $C(1,8)$ and $D(-9,2)$.

4. How far is the point (3,4) from the origin? How far is the point (5,12) from the origin?

5. Let us now find a formula that will describe the length of a line segment determined by any two points in the coordinate plane.



Let the two points be represented by $A(x_1,y_1)$ and $B(x_2,y_2)$. Through A we draw a line parallel to the x-axis. Through B we draw a line parallel to the y-axis. You know that the lines will intersect and form a right angle. Are (x_2,y_1) the coordinates of C, the point of intersection? Are $|x_2 - x_1|$ and $|y_2 - y_1|$ the lengths of the legs of the right triangle? Call the desired length (that is, the length of the hypotenuse) d. Then, by the Pythagorean rule:

$$d^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

and

$$d = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2}$$

But we do not need to use absolute value, since $|x_2 - x_1|^2$ equals $(x_2 - x_1)^2$, and $|y_2 - y_1|^2$ equals $(y_2 - y_1)^2$. Thus we can write:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

This is the **distance formula.** It describes the length of a line segment between any two points (x_1,y_1) and (x_2,y_2) in the coordinate plane.

6. Now let us use the formula to find the distance between the points (5,1) and (2,−3) without drawing a diagram.

Substituting in the formula, we have

$$d = \sqrt{(5-2)^2 + [1-(-3)]^2}, \text{ or } \sqrt{3^2 + 4^2}$$
$$= \sqrt{25}, \text{ or } 5$$

Can we also substitute as follows?

$$d = \sqrt{(2-5)^2 + (-3-1)^2}$$
$$= \sqrt{(-3)^2 + (-4)^2}$$
$$= \sqrt{25}, \text{ or } 5$$

Do you get the same answer for both expressions? Then does it matter which point you think of as (x_1,y_1)?

7. Use the distance formula to determine the distance between each of the following pairs of points.

a. (1,7) and (13,2)
b. (2,−3) and (−2,3)
c. (−1,8) and (−10,−4)
d. (11,4) and (−3,15)
e. (2,2) and (3,0)
f. $(1\frac{1}{2},-1)$ and $(2\frac{1}{4},\frac{1}{2})$

EXERCISES

Use the distance formula to find the distance between each of the following pairs of points.

	a	b
1.	(1,7) and (13,2)	(2,−3) and (−1,1)
2.	(−1,8) and (0,4)	(3,0) and (0,4)
3.	(8,3) and (−1,3)	(−4,−1) and (14,1)
4.	(11,4) and (−3,15)	(2,2) and (7,7)
5.	(1.5, −1) and (3.2, −2)	(−3.2, 4) and (3,4.7)
6.	(300,400) and (298,397)	(.1, .2) and (.01,−.02)

7. The vertices of a square are at $A(0,0)$, $B(3,0)$, $C(3,3)$, and $D(0,3)$. Find the length of AC; of BD. Are the lengths equal?

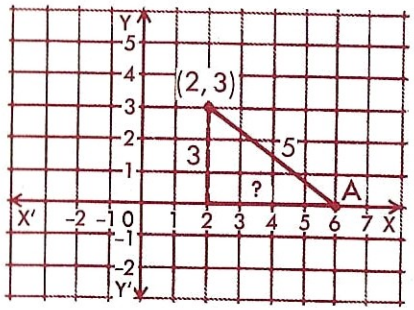
8. The vertices of a rectangle are at $A(-2,3)$, $B(2,3)$, $C(2,-3)$, and $D(-2,-3)$. Show that $mAC = mBD$.

Using the Distance Formula

1. In Exercises **1–6** (page 327), you were given the coordinates of two points and asked to find the distance between these points. If you are given the coordinates of one point and the distance it is from another point, do you suppose you can find the coordinates of the second point?

Let us start by considering this problem.

What point on the x-axis is 5 units distant from the point $(2,3)$?



You might begin by drawing a sketch such as the one shown at the left. Is one leg of the right triangle 3 units in length? Then the length of the other leg of the right triangle is 4 units; that is, $5^2 = 3^2 + 4^2$. What is the x-coordinate of point A? the y-coordinate of point A? Is the point $(6,0)$ 5 units from the point $(2,3)$?

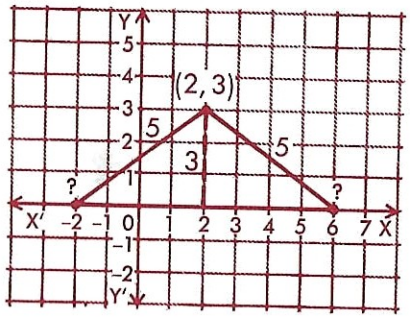
2. Now let us repeat the problem from **1**, using the distance formula. Call the coordinates of point A $(x,0)$. Then:

$$5 = \sqrt{(x-2)^2 + (0-3)^2}$$

Squaring both members: $25 = (x-2)^2 + (0-3)^2$

$$25 = x^2 - 4x + 4 + 9$$

Then:
$$x^2 - 4x - 12 = 0$$
$$(x-6)(x+2) = 0$$
$$x - 6 = 0 \text{ or } x + 2 = 0$$
$$x = 6 \text{ or } x = -2$$



Notice that this method suggests two possible values, 6 and -2, for the x-coordinate of point A. Does each satisfy the original equation? Does this mean that there are two points, $(6,0)$ and $(-2,0)$, that satisfy the conditions of the problem? Did you perhaps think of this when you drew the sketch? If so, your sketch would resemble this one.

3. Find the points on the y-axis that are 10 units distant from the point $(-8,7)$; 10 units from the origin.

4. Draw the line that is the graph of $y = 3$. Find two points on the line that are 13 units from the point $(6,-9)$; 13 units from the origin.

5. Draw the line that is the graph of $x = 5$. Are there two points on this line that are 8 units from the point $(-3,-4)$?

EXERCISES

Use some of the techniques of coordinate geometry that you know to solve the following problems. In each case, plot the points and draw the figure.

1. What is the perimeter of a triangle with the points $A(0,0)$, $B(12,5)$, and $C(6,-3)$ as vertices?

2. Which of the following sets name points that are vertices of isosceles triangles?

a. $(2,5)$, $(-5,-1)$, $(7,-2)$ **b.** $(0,2)$, $(-2,0)$, $(7,-7)$

3. What is the area enclosed by a triangle that has its vertices at $A(0,0)$, $B(7,0)$, and $C(6,4)$?

4. What are the lengths of the diagonals of a quadrilateral that has its vertices at points $(0,0)$, $(8,-2)$, $(8,2)$, and $(-1,9)$?

5. You know that a rhombus is a parallelogram all of whose sides are equal in length. Which of the following sets of points are the vertices of a rhombus? of a square?

a. $(0,0)$, $(3,0)$, $(0,3)$, $(3,3)$ **b.** $(2,0)$, $(3,4)$, $(6,0)$, $(7,4)$

6. Assume that the points $(0,0)$, $(4,0)$, and $(0,3)$ are three vertices of a rectangle. What are the coordinates of the point that is the fourth vertex?

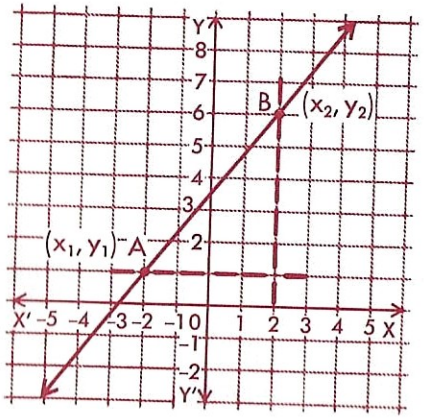
7. Assume that the points $(0,0)$, $(4,0)$, and $(0,3)$ are three vertices of a parallelogram. Describe *three* possible positions for the fourth vertex.

8. Find the lengths of the diagonals of the rectangle in Exercise **6.**

9. Which is the shortest side of a triangle with vertices at points $(0,0)$, $(4,3)$, and $(5,2)$? Which is the longest side?

Equation of a Line

You learned in Chapter 6 how to plot the graph of a linear equation—that is, a graph that is a straight line. How many points determine a straight line? When you drew straight-line graphs (in Chapter 7 as well as in Chapter 6), you started with an equation and determined ordered pairs that satisfied the equation. The set of points corresponding to these ordered pairs was then the graph of the equation (or, stated differently, the graph of the relation whose rule is the equation).



Now let us try to reverse the procedure. Let us start with the coordinates of two points and write an equation whose graph is the line passing through these points. To do this, we will make use of the concept of the slope of a line. You have learned that any line (except a line parallel to the y-axis) has a slope, m. Using any two points on the line, you know:

$$m = \frac{\text{the difference between the } y\text{-coordinates}}{\text{the difference between the } x\text{-coordinates}}$$

Or, in symbols, if $A(x_1,y_1)$ and $B(x_2,y_2)$ are points on the line:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

You have also seen that the line contains only those points whose coordinates are members of the solution set of $y = mx + b$, where m is the slope and b is the y-intercept of the line.

1. Plot points $A(0,3)$ and $B(1,5)$. What is the slope of the line that contains A and B? What is its y-intercept? What is an equation for this line?

2. What is the slope of the line that contains the points $(-1,1)$ and $(1,7)$? We cannot find the y-intercept as easily as in **1**, but we can write $y = 3x + b$. Since we know that $(-1,1)$ is a member of its solution set, replace x by -1 and y by 1. Now solve $1 = 3(-1) + b$. What is an equation for the line containing the points $(-1,1)$ and $(1,7)$?

Find the slope of the line that contains each of the following pairs of points. Then find an equation of the line.

	a	b
3.	$(0,0)$ and $(3,3)$	$(-2,4)$ and $(2,-4)$
4.	$(-1,5)$ and $(5,-1)$	$(0,6)$ and $(2,0)$
5.	$(0,-3)$ and $(6,0)$	$(-1,4)$ and $(3,4)$
6.	$(4,-1)$ and $(5,1)$	$(0,300)$ and $(2,304)$

7. There is another method of finding an equation for a line through two given points. Suppose the points have the coordinates $(-2,4)$ and $(8,1)$.

First find the slope, using the coordinates of the two given points: $\frac{4-1}{-2-8}$

Now express the slope, using any other point (x,y) on the line and a point whose coordinates are known: $\frac{y-4}{x-(-2)}$

Since the slope of a line is everywhere the same, we can write: $\frac{3}{-10} = \frac{y-4}{x+2}$

Then: $3x + 6 = -10y + 40$

$$3x + 10y - 34 = 0$$

How can you check to see that this is a correct equation?

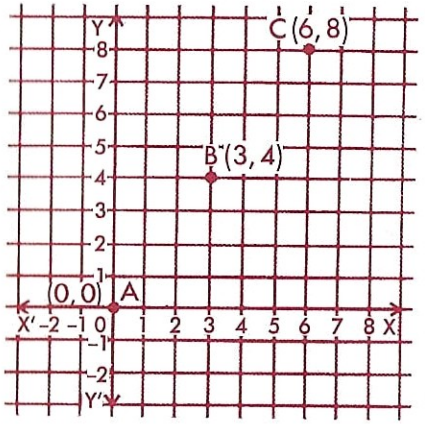
8. Use the method in **7** to find equations for the lines passing through the pairs of points given in **3–6.**

Collinear Points

There are several possible relationships among any three points in the coordinate plane. You know that there is only one point in the coordinate plane corresponding to any given ordered number pair. If points P and Q have the same ordered pair as coordinates, they are called *coincident points;* they coincide. If three points are located so that line segments joining them form a triangle, you know they are not on the same line. Another relationship among a set of points occurs when they lie on the same line. Points that lie on the same line are called **collinear.**

1. Suppose you are given the coordinates of a point P and the slope of a line passing through this point. Can you determine an equation of the line?

What is the slope of the line containing the points $(0,2)$ and $(2,3)$? of the line containing $(2,3)$ and $(6,5)$? Are the slopes equal? Does each line contain the point $(2,3)$? Then can we say that $(0,2)$, $(2,3)$, and $(6,5)$ all lie on the *same* line? The three points are collinear.



2. There are other methods for deciding whether points are collinear. You know that the solution set of a linear equation has a graph in which the points all lie on the same line. Look at the diagram at the left. What is an equation for the line passing through A and B? Now see whether the coordinates of C satisfy the equation. Can you conclude that the points are collinear?

3. In the diagram above, would you expect mAC to equal the sum of mAB and mBC if A, B, and C are on the same line? Check to see whether this is true. Finding the measures of the line segments connecting three or more points offers another way for determining the collinearity of points.

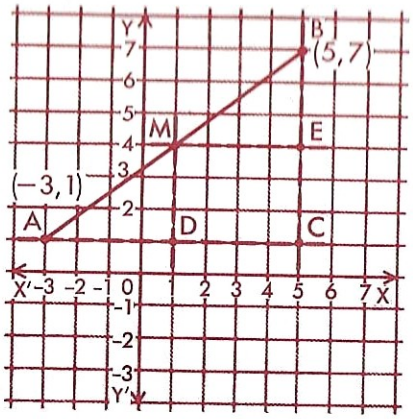
4. Are the three points $P(0,0)$, $Q(2,1)$, and $R(11,5)$ collinear? Test by all three methods.

5. Are the three points $A(-2,2)$, $B(-\frac{1}{2},0)$, and $C(1,-2)$ collinear? Test by all three methods.

Test whether the following sets of points are collinear.

	a	b
6.	$(-1,-1)$, $(0,0)$, $(8,8)$	$(-6,0)$, $(0,-1)$, $(6,1)$
7.	$(1,-3)$, $(0,0)$, $(1,4)$	$(-1,-1)$, $(1,0)$, $(5,2)$
8.	$(-6,-2)$, $(-2,0)$, $(3,3)$	$(101,2)$, $(105,1)$, $(109,0)$

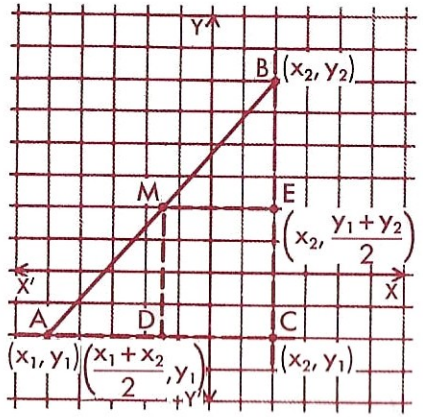
The Midpoint of a Line Segment



1. Let us try to determine the coordinates of the midpoint of an oblique line segment. We will start with two points, A at $(-3,1)$ and B at $(5,7)$. Through A we draw a line parallel to the x-axis; through B, a line parallel to the y-axis. Call the point of intersection of these lines C. What are the coordinates of C? What is the midpoint of line segment AC? Call the point D. What are the coordinates of the midpoint of line segment BC? Call this point E. Now, through D, a line parallel to the y-axis is drawn; through E, a line parallel to the x-axis. These lines intersect at a point we call M. Are $(1,4)$ the coordinates of M?

Does M seem to lie on the line through A and B? Let us prove that it does. Find the slope of the line through $(-3,1)$ and $(5,7)$—that is, the line through A and B. Next find the slope of the line through $(-3,1)$ and $(1,4)$—that is, the line through A and M. Are the slopes equal? Then M lies on AB.

Using the distance formula, find mAM. Find mMB. Are they equal? Then is M the midpoint of segment AB?



2. Let us repeat the construction in 1, starting with any two points, A at (x_1,y_1) and B at (x_2,y_2). The diagram is shown here. What are the coordinates of point C? of D? of E? Then what do you think are the coordinates of M? Do you agree with the following statement?

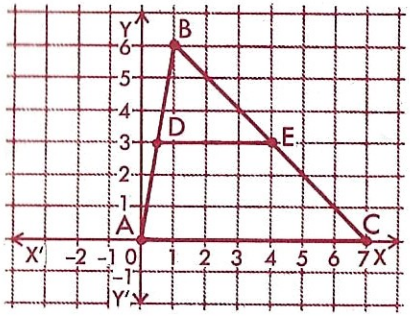
▶ For any two points A at (x_1,y_1) and B at (x_2,y_2) the coordinates of M, the midpoint of AB, are:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

Find the coordinates of the midpoints of the segments whose end points are as follows:

	a	b
3.	(7,10), (1,6)	$(1\frac{1}{2},3)$, (0,−6)
4.	(−8,2), (4,−12)	(7,−8), (15,8)
5.	(13,−5), (−2,3)	(16,3), (7,−18)
6.	(−3,−2), (−15,3)	(1000,500), (−200,−100)

7. The vertices of a triangle are at (0,0), (−2,5), and (6,1). Find the coordinates of the midpoints of the sides of the triangle.



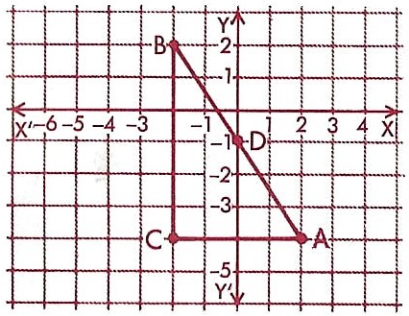
EXERCISES

Examine the figure at the left.

1. Point A is at (0,0), B at (1,6), and C at (7,0). What are the coordinates of D, the midpoint of AB? of E, the midpoint of BC? Now find mAC; mDE. What is the ratio of mDE to mAC?

Repeat Exercise 1 with the coordinates for A, B, and C as follows:

	A	B	C	$mDE:mAC$
2.	(0,0)	(0,6)	(3,4)	?
3.	(−1,−3)	(1,5)	(3,0)	?
4.	(−4,0)	(0,5)	(4,0)	?
5.	(−4,0)	(0,6)	(4,2)	?



What seems to be true of the length of the line segment between the midpoints of two sides of a triangle?

6. Look at the figure at the left. Point C is at (−2,−4), B is at (−2,2), and A is at (2,−4). What are the coordinates of D, the midpoint of AB? Find mCD; mAB. What is the ratio of mCD to mAB?

Repeat Exercise 6 with the coordinates for points A, B, and C as follows:

	A	B	C	$mCD:mAB$
7.	(0,6)	(8,0)	(0,0)	?
8.	(0,−4)	(−4,0)	(0,0)	?
9.	(3,0)	(0,5)	(3,5)	?

10. In each of Exercises **6–9** we considered a triangle with vertices at A, B, and C. The coordinates of the vertices were selected to represent the vertices of right triangles; that is, triangles with a right angle. At which vertex is the right angle in each case? The line segment from a vertex of a triangle to the midpoint of the opposite side is called a **median.** What seems to be true of the relationship between the length of the median drawn to the hypotenuse of a right triangle and the length of the hypotenuse?

Parallel Lines and Perpendicular Lines

1. Suppose the four points A, B, C, and D are located at (0,3), (3,5), (0,1), and (3,3), respectively, and we wish to know whether line AB is parallel to line CD. You know that the slopes of two non-vertical parallel lines are equal. Do lines AB and CD have the same slope?

What is an equation for line AB? for line CD? What is the solution set of this system of equations? Do the lines intersect? Then must the lines be parallel?

▶ If the slopes of two non-vertical lines are equal, the lines are parallel. Of course, all distinct vertical lines are parallel to each other.

In each case, test whether the line containing points A and B is parallel to the line containing points C and D.

	A	B	C	D
2.	(0,0)	(3,4)	(2,1)	(5,5)
3.	(−3,−2)	(0,2)	(−2,1)	(1,5)
4.	(3,5)	(0,−1)	(−4,−2)	(−1,4)
5.	$(3\frac{1}{2},2\frac{1}{4})$	$(3,-1\frac{3}{4})$	(1,4)	$(\frac{1}{2},0)$

6. Using one pair of coordinate axes, draw the graphs of equations **a** and **b** below.

a. $y = 2x$ **b.** $y = -\frac{1}{2}x$

What is the slope of line **a**? the slope of line **b**? Is the product of these slopes -1? Do lines **a** and **b** seem to be perpendicular to each other? Measure the angle with your protractor.

7. Using one pair of coordinate axes, draw the graphs of equations **a** and **b** below.

a. $y = x$ **b.** $y = -x + 3$

Answer the same questions as were given in **6.**

8. Make up a pair of linear equations such that the graphs of these equations will have slopes whose product is -1. Draw the graphs of these equations on one pair of coordinate axes. Are the lines perpendicular?

Does it seem that the following statement is true?

▶ If two lines in the coordinate plane have slopes whose product is -1, the lines are perpendicular lines.

Let us agree to accept this statement and use it as a test for perpendicular lines.

9. Plot point A at $(-1,3)$, B at $(0,0)$, and C at $(6,2)$. Draw AB and BC. What is the slope of line AB? of line BC? What is the product of the two slopes? Then we know that AB is perpendicular to BC.

Suppose A, B, and C have the coordinates shown below. Test whether AB is perpendicular to BC in each case.

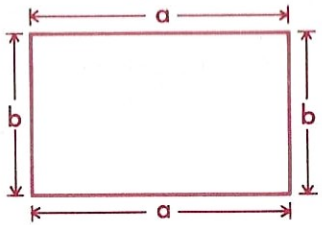
	A	B	C
10.	$(0,0)$	$(3,2)$	$(-2,3)$
11.	$(8,0)$	$(6,4)$	$(-2,0)$
12.	$(6,4)$	$(-2,0)$	$(0,-4)$
13.	$(3,5)$	$(-1,-2)$	$(4,-3)$

Using Algebra to Prove Geometric Relationships

1. Plot four points at $(0,0)$, $(4,0)$, $(0,3)$, and $(4,3)$. Connect the four points with line segments so as to form a quadrilateral. Is the figure formed a rectangle? Why?

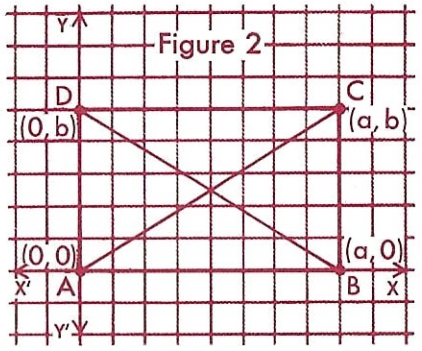
2. Imagine a rectangle whose dimensions are a and b units where a and b are any positive numbers. We can think of this rectangle as *any* rectangle in the set of all rectangles.

Figure 1



Now imagine that we construct our coordinate axes so that the origin coincides with A and the y-axis lies along AD, as shown in Figure 2.

Figure 2



The coordinates of A are then $(0,0)$; the coordinates of B are $(a,0)$; the coordinates of C are (a,b); and the coordinates of D are $(0,b)$.

What are the diagonals of rectangle $ABCD$? Can we express mAC as $\sqrt{(a-0)^2 + (b-0)^2}$, or $\sqrt{a^2 + b^2}$? Express mDB in a similar way. Have we now proved that the diagonals of any rectangle are equal in length?

3. Using Figure 2 above, let us see whether we can prove that the diagonals of any rectangle bisect each other.

Consider diagonal DB. The x-coordinate of its midpoint is $\frac{0+a}{2}$, or $\frac{a}{2}$. The y-coordinate of its midpoint is $\frac{b+0}{2}$, or $\frac{b}{2}$. Thus, the coordinates of the midpoint of DB are $\left(\frac{a}{2}, \frac{b}{2}\right)$. Now consider diagonal AC. Are the coordinates of its midpoint $\left(\frac{0+a}{2}, \frac{0+b}{2}\right)$; that is, $\left(\frac{a}{2}, \frac{b}{2}\right)$?

Do both diagonals have the same midpoint? Then this point is common to both diagonals; that is, it is the point of intersection of the diagonals. Thus, we have proved that the diagonals of any rectangle bisect each other.

EXERCISES

In the following group of exercises, you will need to use five of the skills you have learned involving coordinate geometry. You can:

1. Find the distance between any two given points
2. Find the midpoint of a line segment
3. Find the point of intersection of two lines
4. Write an equation of a line, given two points on it
5. Determine whether two lines are parallel or perpendicular

1. Is the point (1,6) the midpoint of the line segment with end points (3,7) and (−1,5)? If the point (−1,5) is one end point of a line segment and the point (3,4) is the midpoint, what are the coordinates of the other end point?

2. What is the midpoint of the segment that is determined by the two end points (a,b) and (c,d)?

3. Find the midpoint of the line segment between the origin and the intersection of the lines $2x - y = 0$ and $x + y - 9 = 0$.

4. Write an equation for the line that passes through the origin and point (5,−4).

5. Are the points (0,0), (5,0), and (0,7) the vertices of a right triangle? Draw the triangle on a pair of coordinate axes. What are the coordinates of the midpoint of the hypotenuse? Find the distance from this midpoint to each of the three vertices. Leave your answers in radical form. How are the three distances related?

6. Now let us try to prove: In any right triangle, the midpoint of the hypotenuse is equidistant from the three vertices.

Consider a right triangle with one leg b units in length and the other leg a units, where a and b may be any positive numbers. This triangle represents *any* right triangle. Now we construct coordinate axes as shown.

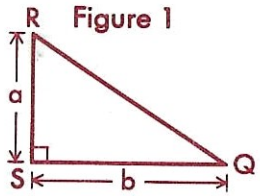


Figure 1

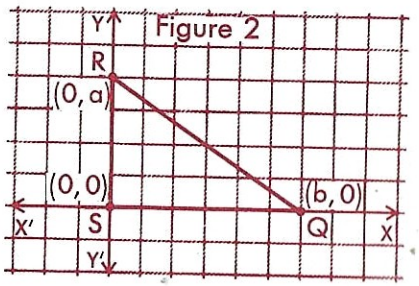


Figure 2

Draw Figure 2 on your paper. What are the coordinates of the midpoint of RQ? Find the distance from this midpoint to each of the three vertices. (Use your work in Exercise **5** as a guide.) Are the distances equal? Do you agree that the midpoint of the hypotenuse in any right triangle is equidistant from the three vertices?

7. Draw a pair of coordinate axes and plot the points (0,0), (8,2), and (4,6). Label them A, B, and C, respectively.

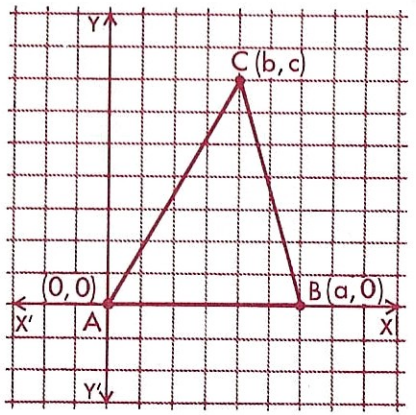
Are the points coincident? collinear? Can they be the vertices of a triangle? Draw the triangle on your coordinate axes.

What is the midpoint of side AC? side BC? Plot and label them D and E, respectively. What is an equation for the line on which side AB lies? What is an equation for the line on which DE lies? Do line AB and line DE have the same slope? Are they parallel?

What is mDE? mAB? (Leave your answers in radical form, but simplify.) How do the two measures compare?

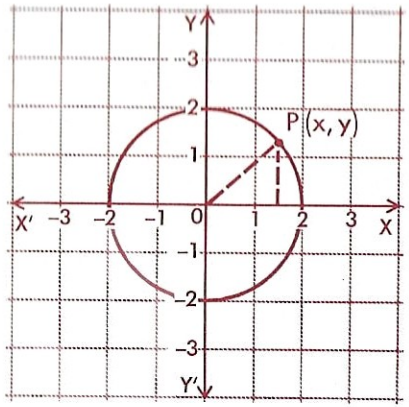
8. Now let us try to prove: The line segment joining the midpoints of two sides of a triangle is half the length of the third side and parallel to the third side. The figure at the right can be used to represent *any* triangle. Notice that we have constructed the coordinate axes so that the x-axis lies along AB.

Follow the procedure of Exercise **7** to complete the proof.



Equation of a Circle

1. You know that the graph of the solution set of an equation is a point or a set of points in the coordinate plane. We call the set of all points that satisfy a given condition (such as an equation) a **locus.** For example, the locus of $x = 2$ is the set of all points that have 2 as their x-coordinate. Therefore, the locus of $x = 2$ is a vertical line passing through (2,0). What is the locus of the condition $x = y$? $y = 0$?



2. You know that a circle is the set of all points in a plane that are at a given distance from a given point in the same plane. Thus, a circle can be described as a *locus* consisting of all points in a plane that satisfy the condition: Each point is at a given distance from a given point in the same plane.

Suppose the given point is the origin. Give the coordinates of a point 2 units from the origin.

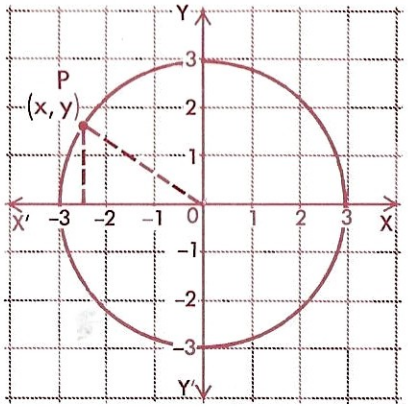
Is $(2,0)$ such a point? $(0,-2)$? $(1,\sqrt{3})$? $(\sqrt{2},\sqrt{2})$? $(-\sqrt{2},\sqrt{2})$? $(\sqrt{3},1)$? Is there an unlimited number of such points?

3. If P is any point (x,y) on a circle with the origin as center and a radius of 2 units, we can use the distance formula to obtain an equation of the circle. Thus, $2 = \sqrt{(x-0)^2 + (y-0)^2}$. Or:

$$2 = \sqrt{x^2 + y^2}$$

Then: $$4 = x^2 + y^2$$

Any points with coordinates that satisfy this equation lie on the circle. Will any points with coordinates that do not satisfy the equation be on the circle?



4. Consider the equation:

$$\sqrt{(x-0)^2 + (y-0)^2} = 3.$$

It states that a point (x,y) is 3 units from the point $(0,0)$. Will any point (x,y) that satisfies this equation be 3 units from the origin? All points that are 3 units from the origin lie on a circle of radius 3. Would this equation, then, be an equation of a circle of radius 3 and center at the origin? Then is $x^2 + y^2 = 9$ an equation of this circle?

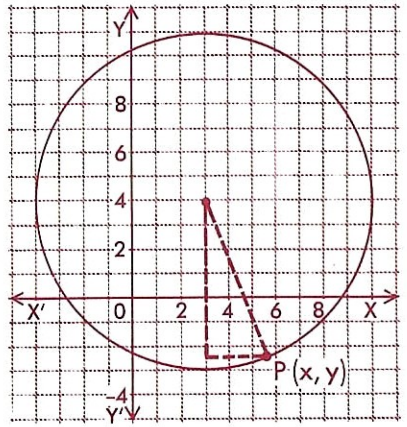
▶ The equation $x^2 + y^2 = r^2$ is an equation of a circle with its center at the origin and a radius of r.

5. What is an equation of a circle with the center at the origin and a radius of 7 units? 4 units? $\frac{1}{2}$ unit?

6. In the equation $x^2 + y^2 = 36$, what is the radius of the circle? Where is the center? Will this circle intersect the line that is the graph of $x = 5$? $y = 7$? $x = 5.999$?

7. In the equation $x^2 + y^2 = 12$, what is the radius of the circle described? Will this circle intersect the line that is the graph of $x = 4$? $y = 3$? $x = 3.4$? $y = 3.5$?

8. Let us try to find an equation of a circle that does *not* have its center at the origin. Look at the graph of a circle at the right. The center is at point $(3,4)$, and the radius is 7. Suppose P with coordinates (x,y) is any point on the circle. Using the distance formula, we can write:



$$7 = \sqrt{(x-3)^2 + (y-4)^2}$$

or: $$49 = (x-3)^2 + (y-4)^2$$

Will this equation be true if x and y are replaced by the coordinates of any point on the circle? Will the coordinates of any point not on the circle satisfy the equation?

▶ If the point (h,k) is the center of a circle with radius r, and point (x,y) is any point on the circle, an equation of the circle is:

$$r^2 = (x-h)^2 + (y-k)^2$$

9. The equation $81 = (x-4)^2 + (y-7)^2$ is the equation of a circle. What is the radius of this circle? Where is the center located? Does this circle intersect the x-axis? the y-axis?

10. The equation $81 = (x+4)^2 + (y+7)^2$ is an equation of a circle. What is the radius? Where is the center located? Does this circle intersect the x-axis? Does it pass through the origin?

EXERCISES

Write an equation for each of the circles described below. Then draw each circle.

1. Radius 4 units, center at (0,0)
2. Radius 1 unit, center at (0,0)
3. Radius $\frac{1}{3}$ unit, center at (0,0)
4. Radius 2 units, center at (1,1)
5. Radius 4 units, center at (5,6)
6. The equation $25 = (x + 3)^2 + (y - 4)^2$ is an equation of a circle. Where is the center of this circle located? Will this circle intersect the x-axis? Will it pass through the origin?

STRUCTURE OF A MATHEMATICAL SYSTEM

Throughout this chapter we have been investigating geometric figures. We have used the coordinate plane to discover some relationships that exist among certain figures. Let us now look at the structure of geometry itself. That is, in a limited way, let us investigate some basic ideas that relate to our earlier work with mathematical systems. In doing this, it will also be helpful to look back at work done in earlier chapters.

Definitions

As you know, it is important in mathematics to specify (whenever possible) a precise meaning for words and symbols that are used. There are various ways of defining words and symbols. But no matter which method is used, a **definition** may be regarded as an agreement concerning the meaning of symbols or words. This agreement, it should be noted, must be formed in such a way that it includes the thing being defined and excludes all other things.

1. One type of definition is used when a new symbol is introduced. For instance, on page 321 we introduced the symbol mRP and said that mRP would mean "the measure of line segment RP." This *defines* the symbol, or tells its meaning.

Define each of the following symbols.

a. $+$ **b.** $\oplus$ in modulo 3 **c.** $\emptyset$ **d.** (x,y)

2. In defining *monomial* on page 122, we used a second type of definition. As you can see by looking back at that page, we defined a monomial by *listing* all the things that could be a monomial in mathematics.

Was *expression* defined by such a listing (page 31)? Was *term* also defined by such a listing (page 106)?

Is every monomial a term? Is every term a monomial?

Is every monomial an expression? Is every expression a monomial?

3. A third type of definition is used when we identify the set to which an object belongs and then specify how the object (or subset of objects) differs from other members of the set. For instance, look at Exercise **5** on page 329. There the text states informally that a rhombus is a "parallelogram all of whose sides are equal in length."

Notice that first the rhombus is identified as a parallelogram, since all rhombuses are members of the set of parallelograms. Next we specify that all of the sides of a rhombus must be equal in length.

Any parallelogram that meets this condition *is* a rhombus, and any parallelogram that does not meet this condition *is not* a rhombus. Are all squares rhombuses? Why is it not true that all rectangles are rhombuses?

4. No matter what form of definition is used, a definition must be *reversible* if it is to be a correct definition. As an example, let us rephrase the definition of a rhombus and study it further. Consider the following.

A. If a figure is a rhombus, then it is a parallelogram all of whose sides are equal in length.

B. If a parallelogram has all of its sides equal in length, then it is a rhombus.

Is statement A true? statement B? When we "reverse" the *if* and *then* parts of a statement (as we reverse A to obtain B), we call the new statement the **converse** of the first. Of course, by reversing B, we obtain A. So statement A is also the *converse* of B. That is, the two statements are converses of each other.

A definition that includes two such statements is sometimes called a **complete definition.** Each of the two statements must be true in a definition.

5. Below is a definition that is familiar to you.

> If an angle has a measure greater than 0° and less than 90°, then it is an acute angle.

What is the converse of this statement? State the definition of an acute angle as a complete definition.

6. A complete definition can be given more simply, using the "if and only if" phrase discussed on page 292 in Chapter 9. For example, we can give a complete definition of a rhombus as follows:

> A figure is a rhombus if and only if it is a parallelogram with all of its sides equal in length.

State the definition of an acute angle in this form.

Although definitions are often stated in a form that does not make use of "if . . . then . . ." or "if and only if," it is important to remember that a definition involves the relationships we have just described.

EXERCISES

Study the list of definitions below.

I. In geometry, a **figure** is any set of points.
II. A **ray** is a figure consisting of a point on a line and all the points extending in one direction from this point.
III. An **angle** is a figure consisting of two rays with a common end point. This common end point is called the **vertex** of the angle.
IV. If two angles have a common vertex and a common ray between their other rays, they are **adjacent** angles.
V. A **line segment** consists of two points on a line and all the points lying between these points.
VI. A **triangle** is a figure consisting of three non-collinear points and the line segments determined by these points.
VII. Each of the segments that form a triangle is called a **side** of the triangle.
VIII. If a triangle has all of its sides equal in length, it is an **equilateral** triangle.
IX. If two lines have a point in common, they are **intersecting lines.**
X. If two lines in a plane do not intersect, they are **parallel lines.**

1. What is the converse of statement IV? Give the complete definition of adjacent angles, using the "if and only if" phraseology.

2. What is the converse of statement VIII? Give a complete definition of an equilateral triangle.

3. Give a complete definition of intersecting lines.

4. Give a complete definition of parallel lines.

5. Try to rephrase statement II in an "if . . . then . . ." form, and give the converse of your sentence. Are both the sentence and its converse true?

6. Try to rephrase statement VI in an "if . . . then . . ." form and give the converse of your sentence. Are both the sentence and its converse true?

7. Try to construct a complete definition of an isosceles triangle; a complete definition of a square.

8. As you know, the converses in a definition are both true. However, with a true statement that is not a definition, the converse may or may not be true. Consider statement *A* and its converse *B*.

A. If a triangle is equilateral, it is isosceles.
B. If a triangle is isosceles, it is equilateral.

You know that all equilateral triangles are isosceles triangles. Thus, statement *A* is true. But if a triangle is isosceles (has two sides equal in length), must it necessarily be equilateral? Then statement *B* is false.

9. None of the following statements is a definition. In each case, tell whether the statement is true. Then write the converse of each and tell whether the converse is true or whether it is false.

a. If two angles are right angles, then their measures are equal.
b. If a number is a member of the set {10, 20, 30}, then it is an even number.
c. If a number is not less than 0 and not greater than 0, then the number is 0.
d. If a figure is a square, then it is a rectangle.

Undefined Words

Look back at your work with definitions and the exercises you have just done. Do you notice how a definition makes use of words previously defined? For instance, to define adjacent angles, we needed an earlier definition of an angle; to define an angle, we needed an earlier definition of a ray.

The word *ray* makes use of the words *point* and *line*. Notice how often the words *point* and *line* occur in the definitions. If you are asked, you can easily draw a model of a point or a line. But the words *point* and *line* cannot really be defined in a mathematically useful sense.

In geometry today, *point* and *line* (and *plane* and a few other such words) are accepted as **undefined words.** We know what we mean by such words, but we do not attempt to *define* them.

If we did attempt such definitions, we would have to define other terms used in these definitions; and then, in defining these terms, we would use still other terms that had to be defined. Or, eventually, we would find that we had traveled "in a circle" and were building our definitions by using words that were themselves defined in terms of *point* and *line*.

Thus, in building geometry as a system, we begin with certain undefined words and construct useful definitions.

If you think back over your mathematics in earlier years, you will find that the word *number* has not been defined. *Number* is an undefined word in our study of systems in algebra.

EXERCISES

1. Over past centuries, various definitions of *point* have been offered. For instance: A *point* is that which has not parts or extent.

Logically, if we use such a definition, we need to have a definition of *parts* and *extent*. Use a dictionary and try to establish a chain of definitions that seems mathematically useful. If you find yourself using a word such as *line* or *point* or *length*, you will know your chain is "circular."

2. In one dictionary, if you wished to find a definition of the word *warmth*, you would obtain the following chain of definitions.

(1) *warmth:* being warm
(2) *warm:* more hot than cold
(3) *hot:* much warmer than the body
(4) *cold:* much less warm than the body

Comment on this chain of definitions.

Axioms

1. Look at this statement. What is the name of this principle? Did we in our work ever *prove* that this statement was true? Did we accept it without proof (and then use it to perform operations in the set of real numbers)?

For all real numbers a and b,
$a + b = b + a$.

2. Throughout your work in geometry in earlier courses—and in your work with graphs in this text—you have often been asked, "How many points are necessary to determine a (straight) line?" And you have answered, "Two." In your answer and in your work in drawing models of lines and line segments, you have accepted without proof the following statement as true:

Two points determine one, and only one, straight line.

Perhaps one reason you accepted the statement was that it was consistent with your experiences in drawing models of lines. Were you always able to draw a line through two points? Did you ever have reason to feel that there was more than one straight line that could be drawn through two points?

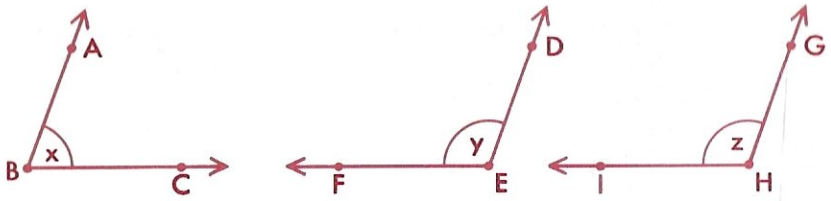
3. Just as we needed some undefined words to serve as the foundation of a system, we must accept some unproved statements as true. These unproved statements—assumptions—deal with fundamental relationships among the elements in a system. We call these the assumed **principles** or **axioms** on which the system is based. Using these basic principles (which generally are consistent with our experiences), we proceed to investigate and prove other relationships.

4. Look back at page 263. There—for the set of real numbers—did we accept the commutative-group properties for addition as principles or axioms in constructing our system? What other principles did we accept?

Look back at pages 132 and 154. What principles did we accept in the set of polynomials under addition? under multiplication? Did we accept as a principle that in the set of polynomials every member has a multiplicative inverse? We rejected this principle because accepting it would have led to a contradiction in our definition of what constitutes a polynomial.

Theorems

1. Suppose it is given that two angles, $\angle ABC$ and $\angle DEF$, are supplementary; also, that $\angle ABC$ and another angle, $\angle GHI$, are supplementary. Do you think that $\angle DEF$ and $\angle GHI$ must be equal in measure? For convenience, let us denote the measure (in degrees) of $\angle ABC$ by x; the measure of $\angle DEF$ by y; the measure of $\angle GHI$ by z.



(1) If two angles are supplementary, the sum of their measures (in degrees) is 180. This is the definition of supplementary angles.

(2) Is it given that $\angle ABC$ and $\angle DEF$ are supplementary? Then $x + y = 180$.

(3) Is it given that $\angle ABC$ and $\angle GHI$ are supplementary? Then what does $x + z$ equal?

(4) You know that when two expressions are connected by an "=" it means that they are names for the same thing. Then is the following true?

$$x + y = x + z$$

(5) Look back at the equation-solving principles stated on page 292. Which principle now permits us to conclude that:

$$y = z$$

Thus, we have shown that the measures of $\angle DEF$ and $\angle GHI$ are equal.

We have now proved: *If two angles are supplementary to the same angle, then they are equal in measure to each other.*

This true statement is not a definition and is not an axiom. We call such a statement a **theorem**, and the procedure we followed is called a *proof* of the theorem. Notice that we constructed our proof on the basis of definitions and assumed principles (such as the equation-solving principles). Once a theorem has been proved, it can then be used to prove other theorems.

2. Let us consider another illustration. Suppose we wish to prove the following: Any relation $\{(x,y) \mid y = kx^2\}$, where k is a constant, is a function. The domain is the set of real numbers.

By definition, a function is a set of ordered pairs such that for each first member in a pair there is a unique second member. Then, if we can prove that for each replacement for x there is a unique value for y, we will have proved that the relation $\{(x,y) \mid y = kx^2\}$ is a function.

Which assumed principle in our work with real numbers can be stated as follows?

If a and b are real numbers, then $a \cdot b$ is a unique real number.

Let us use this principle. If x is replaced by any number, is x^2 a unique real number? Since x^2 is a real number and k is a real number, must it be true that kx^2 is a unique real number?

But $y = kx^2$ is the rule for the relation. Can we then conclude that the relation is a function? Explain.

3. Look back at pages 37-39. What definition was used in the theorem about the sum of two even numbers? which assumed principles? Did we use this theorem in proving the theorem about the sum of an odd number and an even number?

Summary

Throughout the text, we have used the idea of a mathematical system. We have seen that a mathematical system consists of elements and one or more operations. These operations relate the elements in a system.

In the seven pages you have just completed, we have looked at the underlying structure of a system. We have seen that a mathematical system is built on:

(1) Undefined words (These may be the words naming the elements of the system.)

(2) Definitions constructed from undefined words

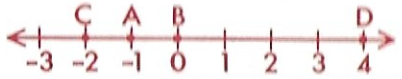
(3) A set of assumed principles or axioms (These are often assumptions about the relationships among the elements of the system.)

(4) Theorems (These are statements of relationships, proved by use of definitions, assumed principles, and so on.)

In your future work in mathematics the structure of a system will be of continuing importance.

Extra Practice page 408

1. Find mAC; mAD; mBC; mDC.



320–321

2. In the coordinate plane, what is the length of each of the line segments with these end points? 323–324

a. (0,0) and (0,5) **b.** $(-\frac{1}{2},0)$ and $(\frac{1}{2},0)$

c. (2,1) and (2,5) **d.** $(1\frac{1}{2},1\frac{1}{2})$ and $(2,1\frac{1}{2})$

Challenge page 423

3. Find the coordinates of the midpoint of each line segment in **2.** 323–324

4. Using the distance formula, find the distance between each of these pairs of points. Answers may be left in radical form. 325–327

a. (2,3) and (8,11) **b.** (0,0) and (1,1)

c. (−1,−1) and (1,2) **d.** (−5,−6) and (0,6)

5. Assume that the points (0,0), (−3,0), and (−3,3) are three vertices of a rectangle. What are the coordinates of the point that is the fourth vertex? 328–329

6. Find the slope of the line through each of these pairs of points.

a. (0,0) and (−4,4) **b.** (−1,3) and (3,−1) 330

7. Are the points (1,2), (2,5), and (7,23) collinear? Test by any method you wish. 331–332

8. Find the midpoint of the line segment between $A(-3,-5)$ and $B(7,9)$. 333–334

9. Is the line through $(1\frac{2}{5},7)$ and (0,0) parallel to the line through (−10,3) and (−15,4)? If not, what relationship exists between them? Give a reason for your answer. 335–336

10. Plot the points (0,0), (5,0), (8,4), and (3,4). Then connect the points by line segments. What kind of quadrilateral is formed? Prove that the diagonals bisect each other. 336–339

11. Write an equation of a circle with radius 6 units and center at origin; radius 2 units and center at point (3,4). 339–341

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

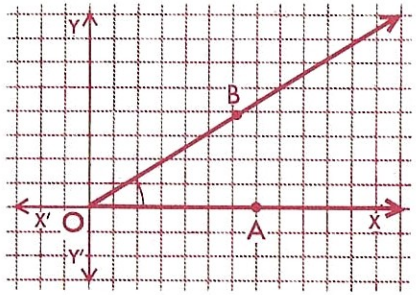
11 Trigonometry

The word *trigonometry* and its meaning (*triangle measurement*) are probably familiar to you. In an earlier course in mathematics, you may have used your knowledge of similar triangles to develop various trigonometric relationships. In this chapter we will make use of the coordinate plane and study (from a somewhat different viewpoint) relationships involved in trigonometry.

THE TRIGONOMETRIC FUNCTIONS

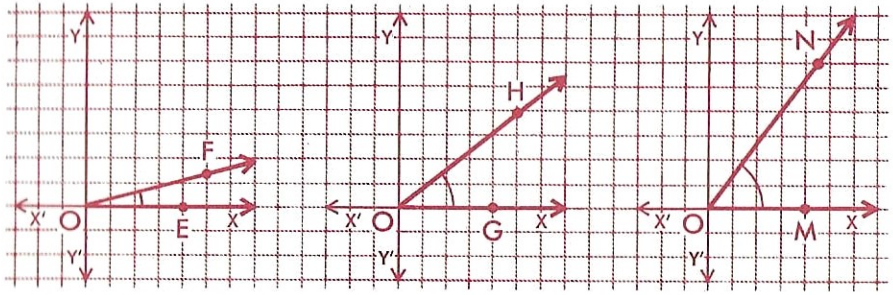
Angle in Standard Position

Let us begin by considering an angle AOB with its vertex at the origin and one side (OA) a ray that extends to the right along the x-axis. This part of the x-axis is usually referred to as the **positive x-axis.**



When an angle has its vertex at the origin and one side lies on the positive x-axis, we say that the angle is in **standard position.**

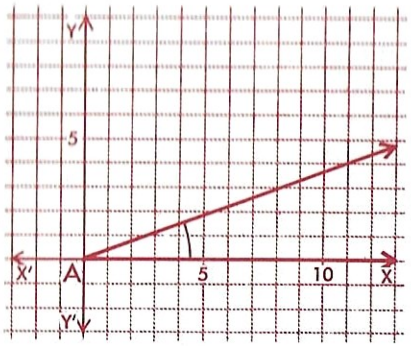
1. Are each of the angles below in standard position? Using your protractor, find the measure of each of the angles.



For convenience, let us agree: In speaking of an angle in standard position, we will call the ray that is the positive x-axis the **initial side** of the angle. The other ray will be called the **terminal side** of the angle.

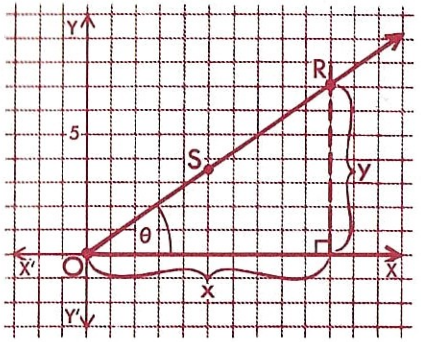
2. Name the initial side of $\angle EOF$; $\angle GOH$; $\angle MON$.

3. Name the terminal side of $\angle EOF$; $\angle GOH$; $\angle MON$.



4. The use of three letters to name an angle can sometimes be cumbersome. Therefore, in working with angles in the coordinate plane, we will often use a single letter such as A, or the Greek letter θ (theta) to name angles. This diagram shows angle A in standard position.

Using your protractor, find the measure of $\angle A$.



Slope of the Terminal Side

1. This diagram shows an angle θ, of measure 35°, in standard position. From point R on the terminal side a perpendicular is drawn to the x-axis.

Are (0,0) and (10,7) the coordinates of two points on the terminal side? What is the slope of the line containing OR? Let us refer to this slope as the slope of the terminal side of θ.

Suppose you had chosen point S instead of R to use in determining the slope of the terminal side. Would you have obtained the same answer?

2. You have found that the slope of the terminal side of θ (in **1**) is $\frac{7}{10}$ or .7. Then will the following be true for any point $P(x_1,y_1)$, except the origin, on the terminal side?

$$\frac{y_1}{x_1} = .7$$

Can you use this relationship to determine the ordinate of a point whose abscissa is given? For instance, if x_1 is 20, then:

$$\frac{y_1}{20} = .7$$

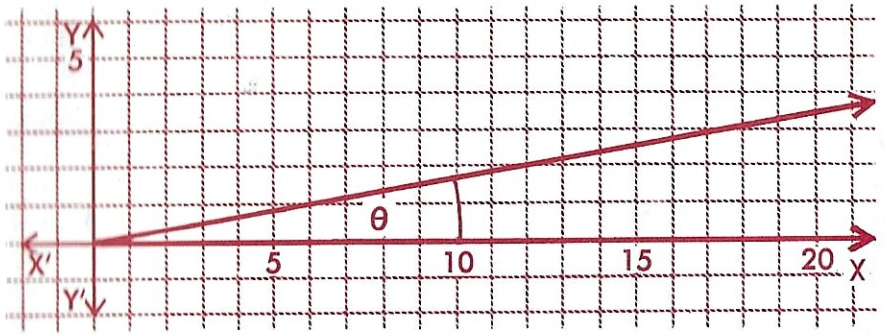
Is the ordinate 14 in this case?

3. Point A is on ray OR and has abscissa 1000. What is the ordinate?

4. Point B is on ray OR and has abscissa 2. What is the ordinate?

5. Below is an angle θ of 10° in standard position. What is the slope of the terminal side? Express the slope as a decimal to the nearest hundredth.

Did you obtain one of these numbers: .16, .17, or .18? Let us accept .18 as the result and use it for the work that follows.



6. Will it be true that for any point $P(x_1,y_1)$ on the terminal side of $\angle\theta$, $\frac{y_1}{x_1} = .18$? Use this relationship to determine the approximate ordinate of a point on the terminal side:

a. if the abscissa is 14
b. if the abscissa is 23
c. if the abscissa is 350

The Tangent Function

1. On a pair of coordinate axes, draw an angle with 20° measure in standard position. Pick a point on the terminal side of the angle and use its coordinates to determine the slope of the terminal side. Express the slope as a decimal to the nearest hundredth.

2. On a pair of coordinate axes, draw an angle with 30° measure in standard position. Follow a procedure similar to that described in **1** and find the slope of the terminal side (to the nearest hundredth).

3. Repeat the procedure described in **1** for an angle with measure 40°; 50°; 60°; 70°.

4. Copy and complete the table that is shown below.

For an angle with measure:	10°	20°	30°	40°	50°	60°	70°
Slope of terminal side of angle is:	.18						

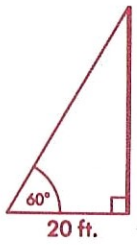
Can you think of your results as a set of ordered pairs? Does it seem that associated with each first member (each angle measure) there is exactly one second member? Then do you believe that this relationship is a function? It is, in fact, a function.

The **tangent function** is the set of all ordered pairs in which the first member is the measure of an angle in standard position and the second member is a number equal to the slope of the terminal side of the angle. In each case, the number is called the **tangent** of the angle to which it corresponds. Thus, (to the nearest hundredth) .36 is the tangent of an angle whose measure is 20°. This statement is written in abbreviated form as:

$$\tan 20° = .36$$

(read "tangent of 20° = .36")

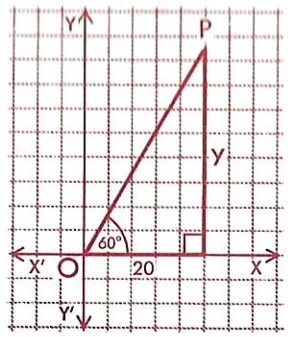
Using the Tangent Function



1. The tangent function is useful in solving certain kinds of problems. For example, suppose a guy wire from the top of a pole makes an angle of 60° with the ground and is fastened to the ground at a distance of 20 feet from the pole. Can we now find the height of the pole?

In approaching such a problem, we make certain assumptions about the physical situation that may not be entirely accurate but *are* a reasonably good description of the real situation. In this case, we assume that the pole is vertical (makes a right angle with the ground), and that the pole, the rope, and the ground can be represented as "straight lines." This is indicated in the figure above.

Now imagine a coordinate system with the x-axis along the ground and the origin at the point at which the wire meets the ground. Using what we have learned about slope and the tangent function, we write:

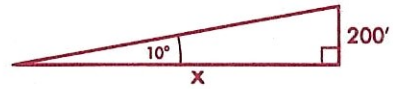


$$\tan 60° = \frac{y}{20}$$

Your completed table shows that tan 60° is 1.73. Then: $1.73 = \frac{y}{20}$

Solve for y. What is the height of the pole (to the nearest foot)?

2. Suppose you know that the height of a certain tree is 200 feet, and the measure of the angle made by the ground and a line to the top of the tree is 10°. How far are you from the tree? If x stands for the number of feet you are from the tree, is it true that: $\tan 10° = \frac{200}{x}$? (You may wish to draw a sketch of your own, with the coordinate axes placed so that the 10° angle is in standard position.)

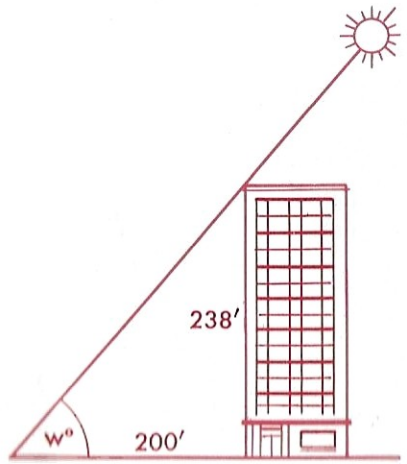


From your table, you can see that tan 10° is .18. Therefore, we have:

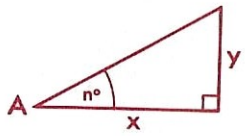
$$.18 = \frac{200}{x}$$

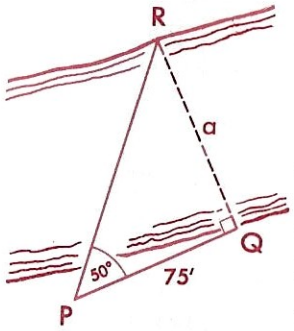
Solve for x. How many feet are you from the tree?

3. A certain building is 238 feet tall. It casts a shadow 200 feet long. What angle does a ray from the sun make with the ground? The problem can be visualized as in the picture at the right. If we let w stand for the number of degrees in the angle, is it true that $\tan w° = \frac{238}{200}$? Then $\tan w° = 1.19$. From your table, what is the measure of an angle which has a tangent equal to 1.19?



Look back at your work in **1–3** on pages 354–355. In each case, is the problem visualized in terms of a right triangle? Notice that each involved the measure of an acute angle; the measure of the side opposite this angle; and the measure of the side adjacent to the angle. Do you see that when we know two of these measures, we can use the tangent function to find the third number involved in the equation?





4. Two trees are located at Q and R on opposite banks of the river. A distance of 75 feet is measured at right angles to RQ. The measure of angle P is found to be 50°. To find the approximate width of the river (the distance between R and Q), we can use the tangent function. Write the equation. Is your equation $\tan 50° = \frac{a}{75}$, or is it $\tan 50° = \frac{75}{a}$? Solve for a. What is the approximate width of the river?

We have said that a single letter such as θ is often used to name an angle. It is also common to use θ in a somewhat different way, referring to the tangent of the angle. If θ is an angle whose measure is 40°, "tan θ" and "tan 40°" are used to mean the same thing. They are different ways of indicating the same number.

On page 427 is a table containing (in one of its columns) values of tangents for many more angles than you have in your list. These numbers are correct to four decimal places. For the work with tangents, you will need to use only the columns marked "Angle" and "Tan."

5. If the measure of θ is 22°, what is $\tan \theta$? Did you find it to be .4040? Use the table to find tangents for angles with the following measures:

a. 11° **b.** 74° **c.** 25° **d.** 33° **e.** 87°

6. If .7002 is the tangent of an angle, what is the measure of the angle? Find .7002 in the table and see what angle corresponds to it. Is it true that $\tan 35° = .7002$? Find the measure of the angle which has each of the following numbers as a tangent:

a. 8.1443 **b.** .3640 **c.** 1.4826 **d.** 1.0000 **e.** .2309

7. Suppose $\tan \theta = .1407$, and you wish to find the measure of $\angle\theta$. We can denote this as $m\angle\theta$. Do you find the entry .1407 in the "Tan" column? Find the two numbers in the "Tan" column that are closest to .1407, one smaller than .1407 and one larger than .1407. Which is closer to .1407? Then, to the nearest degree, $\angle\theta$ measures 8°.

Find the measure of $\angle\theta$ to the nearest degree in each of the following.

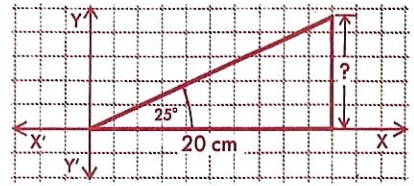
a. $\tan \theta = .7815$ **b.** $\tan \theta = 9.5200$ **c.** $\tan \theta = 2.6603$
d. $\tan \theta = .8900$ **e.** $\tan \theta = .3905$ **f.** $\tan \theta = .9500$

EXERCISES

1. If a line passes through the origin and the point (1,1), what angle does it make with the x-axis?

2. If a line passes through the origin and the point (3,2), what angle does it make with the x-axis? (Express the measure to the nearest degree.)

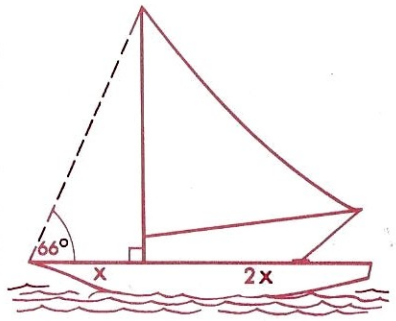
3. If a right triangle has one acute angle of 25° and the side adjacent to this angle is 20 centimeters long, how long is the side opposite the angle?

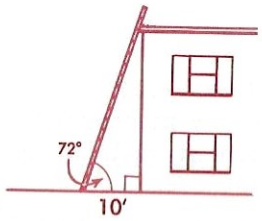


4. If a right triangle has one acute angle of 59°, and the side opposite this angle is 11 inches long, how long is the adjacent side?

5. A right triangle has one side 13 centimeters long and the other side measures 12 centimeters. Find the measures of the two acute angles. How long is the hypotenuse?

6. A 9-foot mast is placed two-thirds of the way forward on a certain sailboat. If a line which runs from the bow to the top of the mast makes an angle of 66° with the deck, how long is the boat?

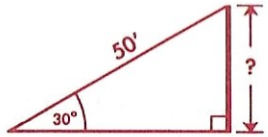




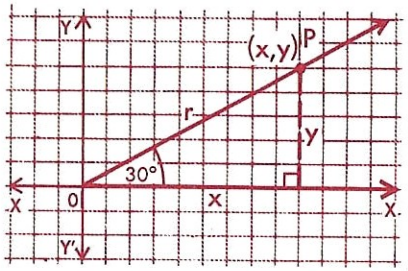
7. a. A ladder, placed 10 feet from the base of a building makes an angle of 72° with the ground when leaned against the top of the building. How tall is the building?

b. If the foot of the ladder is placed 6 feet from the building, what angle does the ladder make with the ground?

The Sine Function



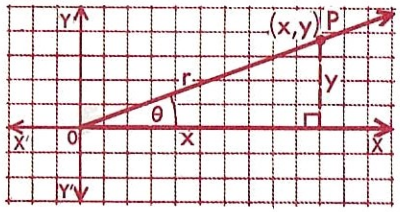
1. A 50-foot rope is stretched tight from the top of a pole to the ground. It makes an angle of 30° with the ground. Can you find the height of the pole, using the tangent function?



Now suppose you know that the following relationship is true: For an angle of 30° in standard position, if we choose a point P on the terminal side of the angle, the ordinate (y) of the point divided by its distance (r) from the origin is always equal to $\frac{1}{2}$.

Could you then visualize the problem using coordinate axes and find the solution by solving the equation: $\frac{1}{2} = \frac{y}{50}$? What would y equal?

2. If the rope were 100 feet long and the angle were 30°, what would the height of the pole be? if the rope were 70 feet long and the angle were 30°?

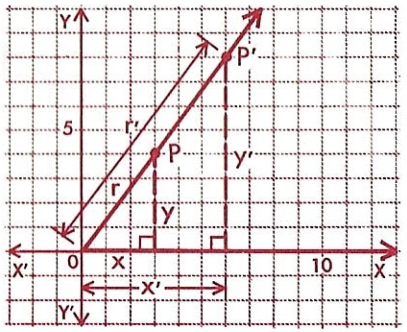


3. Consider an angle θ in standard position and the point $P(x,y)$ located on the terminal side of θ. Using the distance formula, can you express the length of r in terms of x and y?

Does $r = \sqrt{x^2 + y^2}$?

Do you see that, for any point P on the terminal side of $\angle\theta$, there are three related lengths: x, y, and r?

Let us investigate the quotient $\frac{y}{r}$ for a number of angles. Our aim will be to see whether a unique number $\frac{y}{r}$ is associated with a given angle.



4. The figure at the right shows an angle in standard position with its terminal side passing through $P(3,4)$. Use the distance formula to find r, the distance between $(3,4)$ and the origin. What does r equal? Then $\frac{y}{r} = \frac{4}{5}$ or, expressed in hundredths, .80.

Pick another point on the terminal side of the angle—for example, P'. What are the coordinates of P'? Find the distance between $(6,8)$ and the origin. What does $\frac{y}{r}$ equal (expressed in hundredths)?

Have we found that $\frac{y}{r} = \frac{y'}{r'}$?

Using your protractor, find the measure of the angle.

5. a. On coordinate axes, draw an angle in standard position with its terminal side passing through $(2,2)$. Find the distance r between $(2,2)$ and the origin. What does $\frac{y}{r}$ equal for this angle? Express your answer as simply as you can in radical form.

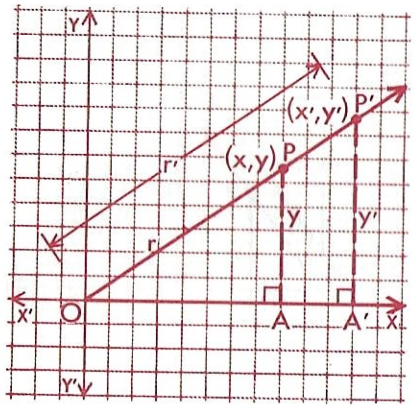
b. Pick another point P' on the terminal side of the angle. What are its coordinates? Find the distance r' between this point and the origin. What does $\frac{y'}{r'}$ equal? Again express your answer as simply as you can in radical form.

Does $\frac{y}{r} = \frac{y'}{r'}$? What is the measure of the angle?

6. a. Repeat the procedure of **4** and **5** for an angle in standard position with its terminal side passing through $(8,15)$.

b. Repeat the procedure of **4** and **5** for an angle in standard position with its terminal side passing through $(5,12)$.

7. Does it seem that for an angle in standard position there is a unique number $\frac{y}{r}$ that does not depend on which point on the terminal side we select for use in our computations? Let us prove this is so. This figure shows an angle O in standard position. From any points P and P' we draw perpendiculars to the x-axis. The points of intersection are called A and A', respectively.



What kind of triangles are $\triangle PAO$ and $\triangle P'A'O$? You can see that $\angle O$ is an angle common to both triangles. Then, from your previous courses in mathematics, you know the angles of $\triangle PAO$ are equal in measure to the corresponding angles in $\triangle P'A'O$, and the triangles are similar.

Since corresponding sides of similar triangles are proportional, can we write:

$$y:y' = r:r'$$

You know that a proportion is true if and only if the product of the *means* equals the product of the *extremes*. Thus:

$$y' \cdot r = y \cdot r'$$

Using equation-solving principles, show that this true statement is equivalent to:

$$\frac{y'}{r'} = \frac{y}{r}$$

This completes our proof.

8. We have seen that the number $\frac{y}{r}$ does not depend on which point on the terminal side we select. We have also observed that to an angle of measure 45° there corresponds the number $\frac{\sqrt{2}}{2}$ (expressed in hundredths, .71); to an angle of measure 53°, there corresponds the number .80; to an angle of measure 62°, the number .88; to an angle of measure 67°, the number .92. Draw an angle of 20° in standard position and express $\frac{y}{r}$ in hundredths. Repeat the procedure for an angle of 35° in standard position.

Do you see that you have obtained a set of ordered pairs? We can now identify another function important in trigonometry.

The **sine function** consists of the set of ordered pairs in which the first member is the measure of an angle in standard position and the second member is the number $\frac{y}{r}$. We call the number $\frac{y}{r}$ the **sine** of the angle. Thus, $\frac{\sqrt{2}}{2}$ or .71 is the sine of an angle whose measure is 45°. This statement is abbreviated: $\sin 45° = .71$.

And, more generally: $\sin \theta = \frac{y}{r}$

EXERCISES

As you may have noticed, the table of values of trigonometric functions on page 427 contains a column marked "Sin." Use this table for all your future work with the sine function.

1. What is the sine of an angle whose measure is 22°? Did you find that it is .3746? Find the sines of angles with the following measures.

a. 75° **b.** 8° **c.** 55° **d.** 30° **e.** 88° **f.** 60°

2. What is the measure of an angle that has a sine equal to .6561? Is the answer 41°? Find the measure of angle θ in each of the following cases.

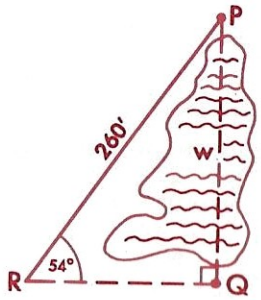
a. $\sin \theta = .2588$ **b.** $\sin \theta = .9925$ **c.** $\sin \theta = .8660$

3. Find, to the nearest degree, the measure of the angle in each of the following cases.

a. $\sin \theta = .6025$ **b.** $\sin \theta = .3750$ **c.** $\sin \theta = .7433$
d. $\sin \theta = .7658$ **e.** $\sin \theta = .9880$ **f.** $\sin \theta = .3418$

Using the Sine Function

Look back at your work in **4–8** on pages 359–360. Does each of the situations where you investigated $\frac{y}{r}$ involve a right triangle? Is y in each case the length of the side opposite the angle? What is r in each case?



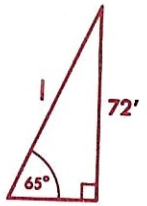
1. To find the distance (w) between two points P and Q across a lake, a surveyor made the measurements indicated in the drawing. Can we write the following? (Think of an angle of 54° in standard position.)

$$\sin 54° = \frac{w}{260}$$

Then, using the table:

$$.8090 = \frac{w}{260}$$

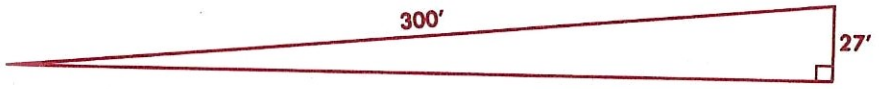
Complete the solution. What is the approximate distance between P and Q?



2. A guy wire for a 72-foot pole makes an angle of 65° with the ground. Find the length of the guy wire (disregarding the length used in fastening the wire).

Is the correct equation $\sin 65° = \frac{l}{72}$, or is it $\sin 65° = \frac{72}{l}$? Complete the solution.

3. The acute angle that an oblique line makes with the horizontal is called its **angle of inclination.** In a drawing, a straight stretch of hill can be represented as an oblique line segment and we speak of the angle of inclination of the hill. This drawing shows a hill that rises 27 feet in a distance of 300 feet. To the nearest degree, what is the angle of inclination of the hill?

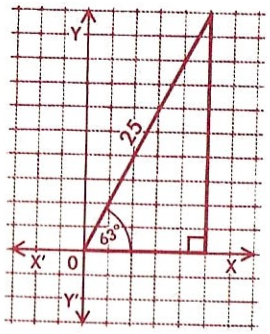


4. A straight stretch of hill has an angle of inclination of 3°. What is the difference in elevation between two points that are 750 feet apart on the hill?

5. A straight stretch of hill has an angle of inclination of 6°. If there is a difference in elevation of 45 feet between a point at the bottom of the hill and a point at the top of the hill, how many feet must a car travel in climbing the hill?

EXERCISES

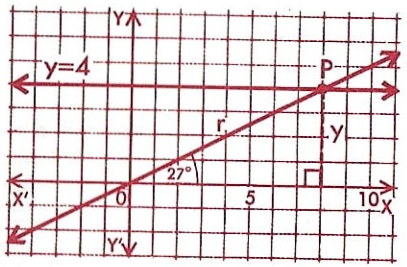
1. If a right triangle has one acute angle measuring 63° and the hypotenuse is 25 inches long, how long is the leg which is opposite the 63° angle?



2. If a right triangle has one acute angle with measure 21° and the leg opposite this angle is 14 centimeters long, how long is the hypotenuse?

3. Planks are laid on the end of a truckbed to form a ramp for a tractor to be loaded on the truck. If the truckbed is $4\frac{1}{2}$ feet high and the greatest angle at which the tractor can safely climb is 40°, what is the minimum length that the planks can be?

4. A line passes through the origin and makes a 27° angle with the x-axis. If it intersects the line that is the graph of $y = 4$ at point P, what is the length of segment OP?



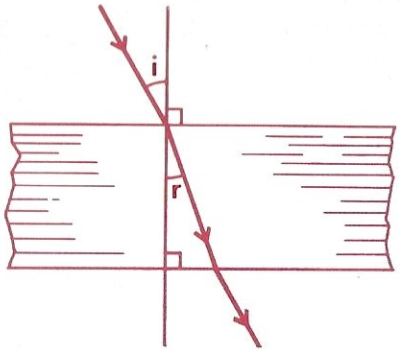
5. A straight stretch of hill rises 75 feet in a distance of 1000 feet measured along the hill. To the nearest degree, what is the angle of inclination of the hill?

6. A 15-foot ladder, leaning against a building, makes an angle of 72° with the ground. How far above the ground is the point where the ladder meets the building?

7. An airplane gains 560 feet in altitude in traveling from R to T, a distance of 3450 feet. To the nearest degree, at what angle is the plane climbing?

Snell's Law

When a light ray passes from one transparent substance to another, the path of the ray is bent. This bending is called **refraction.** The amount of refraction is determined by the material and the angle at which the ray strikes the surface.



In the diagram a light ray passes from air through a piece of glass; i is the measure of the *angle of incidence* and r is the measure of the *angle of refraction.*

In 1621 Willebrord Snell discovered that the ratio $\frac{\sin i}{\sin r}$ is constant for a given substance. For example, for some kinds of glass, $\frac{\sin i}{\sin r} = 1.5$; for diamond, $\frac{\sin i}{\sin r} = 2.42$. In each case, the ratio $\frac{\sin i}{\sin r}$ is called the **index of refraction** of the substance.

1. If i is 30°, what is r for glass?

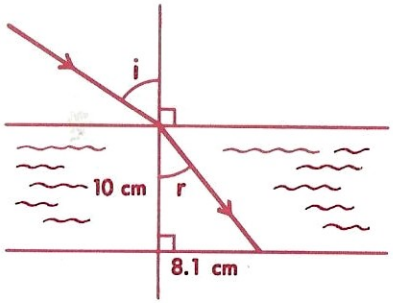
$$\frac{\sin i}{\sin r} = 1.5$$

$$\frac{\sin i}{1.5} = \sin r$$

What is the sine of 30°? Then $\frac{.5}{1.5} = \sin r$. What is r?

2. If i is 39° and r is 20°, what is the index of refraction?

3. If $r = 30°$ and the index of refraction is 1.33, what is i?



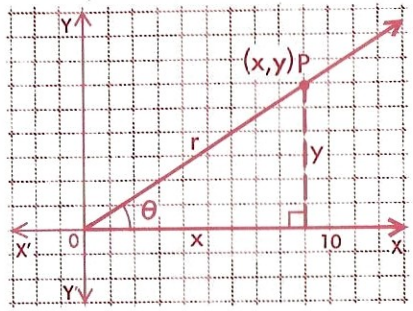
4. A beam of light passes through a pan of water 10 centimeters deep. It strikes the bottom 8.1 centimeters from a point in line with where it entered. The index of refraction for water is 1.33. What is r? (Hint: Use the tangent function to find r.) What is i?

The Cosine Function

You have been working with the tangent function and the sine function—two basic **trigonometric functions.** Now, using this experience, you can investigate on your own a third trigonometric function.

1. Consider an angle θ in standard position with any point $P(x,y)$ on its terminal side.

The **cosine** of angle θ is the number $\frac{x}{r}$; that is, the number obtained by dividing the abscissa of P by its distance (r) from the origin. We write: $\cos\theta = \frac{x}{r}$. (How is tan θ defined? sin θ?)



2. On a pair of coordinate axes, draw in standard position an angle whose terminal side passes through (12,5). Determine r and express $\frac{x}{r}$ in hundredths. Using your protractor, find the measure of the angle.

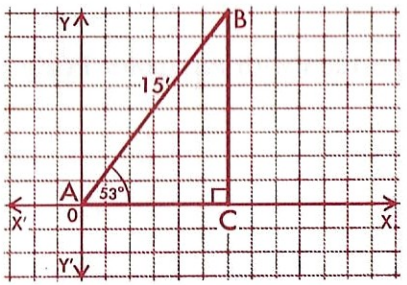
3. Repeat the procedure of **2** for an angle whose terminal side passes through the points named in **a–d.** (In your computations, you may use the table of square roots on page 426.) Find the measure of each angle.

a. (12,6) **b.** (12,9) **c.** (12,12) **d.** (12,16)

4. Make a table pairing the measure of each angle with the number $\frac{x}{r}$ associated with the angle. Do you think that for each angle there is a unique $\frac{x}{r}$?

5. Prove that the number we obtain as $\frac{x}{r}$ does not depend on the point on the terminal side that we select for use in our computations. (Look back at **7** on page 360 for help in your proof.)

Let us now identify the **cosine function.** It is the set of all ordered pairs in which the first member is the measure of an angle in standard position and the second member is the quotient obtained by dividing the abscissa (x) of a point on the terminal side by its distance (r) from the origin.

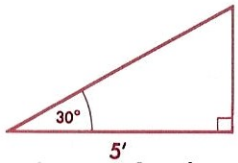


6. Study the entries in the table you made in **4**, page 365. Do you agree that cos 53° = .60 (approximately)? Then, if one angle of right triangle ABC measures 53° and if the hypotenuse of the triangle is 15 feet long, what is the length of side AC?

EXERCISES

The table on page 427 may be used for this work and all future computations with trigonometric functions.

1. If one angle of a right triangle measures 60° and the hypotenuse of the right triangle measures 75 feet, what is the length of the side adjacent to the angle?



2. A brace for a post is to be anchored 5 feet from the base of the post and must make an angle of 30° with the horizontal. How long must the brace be?

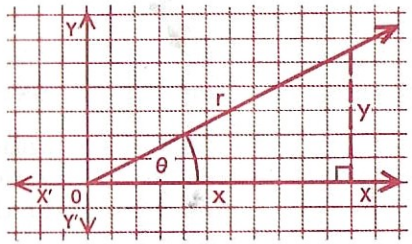
3. One angle of a right triangle measures 43° and the leg adjacent to this angle is 8.6 inches long. How long is the hypotenuse?

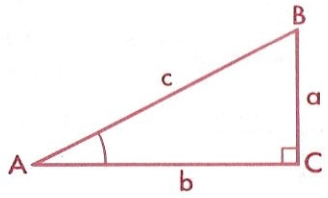
4. One leg of a right triangle is 15 centimeters long and the hypotenuse is 22 centimeters long. Find the measures of both acute angles.

Trigonometry of the Right Triangle

As you have worked with the tangent, sine, and cosine of an acute angle, you have found that the problem situations can be visualized in terms of right triangles. The phrases "length of the side opposite the angle," "length of the side adjacent to the angle," and "length of the hypotenuse" have been used in various exercises.

For an angle in standard position, which phrase can be used to describe y? x? r? Study the figures below and compare the two ways in which the tangent, sine, and cosine of an acute angle are described. You may find it helpful to remember these diagrams.



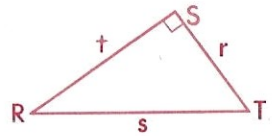


$$\sin\theta = \frac{y}{r} \qquad \sin A = \frac{\text{length of side opposite } A}{\text{length of hypotenuse}} \qquad \left(\frac{a}{c}\right)$$

$$\cos\theta = \frac{x}{r} \qquad \cos A = \frac{\text{length of side adjacent to } A}{\text{length of hypotenuse}} \qquad \left(\frac{b}{c}\right)$$

$$\tan\theta = \frac{y}{x} \qquad \tan A = \frac{\text{length of side opposite } A}{\text{length of side adjacent to } A} \qquad \left(\frac{a}{b}\right)$$

1. This figure shows *any* right triangle RST with S the right angle and r, s, and t the measures of the sides as indicated.



Is it true that $\sin R = \frac{r}{s}$? What does cos T equal? Then is cos T equal to sin R?

Is it true that $\cos R = \frac{t}{s}$? What does sin T equal? Then is sin T equal to cos R?

2. Are angles R and T complementary angles—that is, is the sum of their measures 90°? Then is the sine of an angle equal to the cosine of its complement?

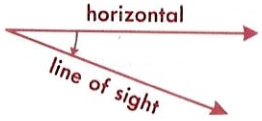
3. Suppose it is given that $\sin 30° = \frac{1}{2}$; $\cos 45° = \frac{\sqrt{2}}{2}$; and $\sin 60° = \frac{\sqrt{3}}{2}$. What is sin 45°? cos 30°? What is the measure of an angle whose cosine is $\frac{1}{2}$?

EXERCISES

In the following problems, use the trigonometric function that you think is best suited to finding the solution. There will often be several methods of solution possible.

When an angle is measured from the horizontal *upward*, it is said to be an **angle of elevation.**

An angle that is measured *downward* from the horizontal is called an **angle of depression.**

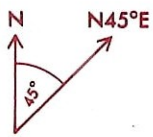


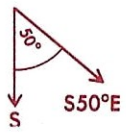
1. An airplane flying at 5000 feet altitude is picked up on radar at an angle of elevation of 15°. What is the ground distance of the airplane from the radar station?

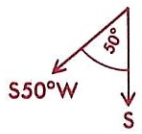
2. Ben flies a kite that he observes to be at an angle of elevation of 40°. For every 10 feet of string that he plays out, how much higher will the kite go?

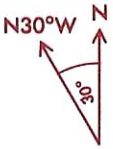
3. From an airplane flying at 20,000 feet, a stadium is observed at an angle of depresson of 66°. How far is the airplane from the stadium?

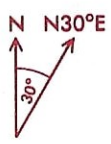
Directions are often given by compass **bearings** or **headings.** These are stated in degrees east or west from north and south. Study the illustrations. A heading of N45°E is called northeast; a heading of S45°W is called southwest.

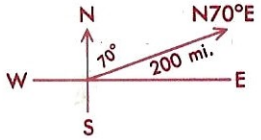












4. A ship sails N70°E for 200 miles. How far eastward has it gone? how far northward?

5. A ship sails N55°E for 350 miles. How far northward has it gone? how far eastward?

6. In order to avoid some shoals, a ship at sea must pass no closer than 250 yards from a certain cliff which is 150 feet high. What is the greatest angle of elevation of the top of the cliff that a pilot of a ship can observe and remain safe?

VECTORS

Ordered Pairs as Vectors

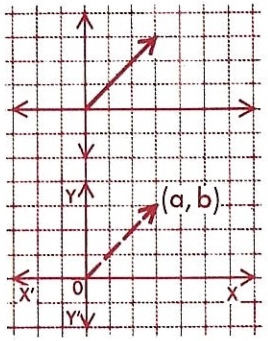
1. As you know, when small distances are involved, we can regard the earth's surface as a plane. Solve the following problems. Use a graphical means when you think it may help you.

How far from your starting point would you be if you went:

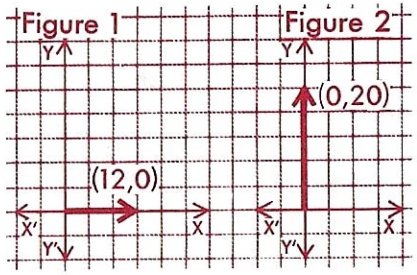
a. 3 miles east and then 1 mile west?
b. 3 miles west and then 2 miles east?
c. 3 miles east and then 2 miles east?
d. 3 miles east and then 4 miles north?
e. 4 miles north and then 3 miles east?
f. 5 miles north and then 2 miles west?
g. 3 miles north and then 3 miles northeast?
h. 5 miles northeast and then 3 miles southeast?
i. 3 miles northeast, then 2 miles east, then 4 miles southwest?

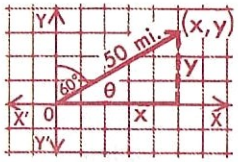
Can you see that problems of this type can become involved and difficult? There is a method, however, whereby they can be systematically solved. This method makes use of the idea of a **vector.**

In physics a vector is often described as a directed line segment. In mathematics, if we think of a vector as having its beginning at the origin in the coordinate plane, we can represent the vector by an ordered pair of numbers—the pair that corresponds to the point at the "head" of the vector. In both instances we are representing *magnitude* (the length of the line segment or the distance of the point from the origin) and *direction.*



2. A move of 12 miles to the east can be shown as in Figure 1. A force of 20 pounds acting toward the north can be shown as in Figure 2. Draw a vector showing a force of 8 pounds acting toward the west; a move of 16 miles to the south. In each case, write the ordered pair next to the head of the vector.





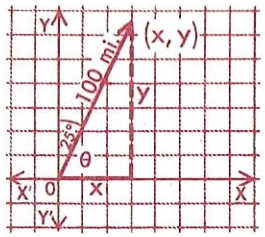
3. The vector in this diagram represents a velocity of 50 miles per hour in the direction N60°E. Notice that in this case we do not know the ordered pair that corresponds to the point at the head of the vector. However, using trigonometry, we can find this ordered pair. Does the measure of $\angle\theta$ equal 30°? Why?

We know that $\cos\theta = \frac{x}{50}$. Then is the following true?

$$x = 50 \cdot \cos\theta$$

Do we know that $\sin\theta = \frac{y}{50}$? Then:

$$y = 50 \cdot \sin\theta$$



Find what $50 \cdot \cos 30°$ equals; $50 \cdot \sin 30°$. Are (43,25) the coordinates of the point at the head of the vector?

4. Shown here is a vector representing a velocity of 100 miles per hour in the direction N25°E. Find the coordinates of the point at the head of the vector.

EXERCISES

Sketch a vector to represent each of the following. Use a coordinate system and name the coordinates of the point that is the head of the vector.

1. A force of 30 pounds acting to the north
2. A force of 35 grams acting to the east
3. A force of 20 pounds acting at a heading of N30°E
4. A velocity of 100 miles per hour northeast
5. A velocity of 250 miles per hour N25°E

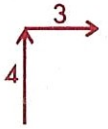
Vector Addition

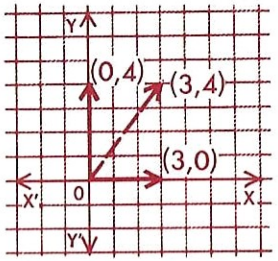
1. Look at your work for **1a** on page 369. Perhaps your answer was "2 miles"—or, even more specifically, "2 miles east."

Now let us simply use number pairs. Does (3,0) represent a movement of 3 miles to the east? Does (−1,0) represent a movement of 1 mile to the west? What number pair corresponds to "2 miles east"? Let us use the form of an equation and write:

$$(3,0) + (-1,0) = (2,0)$$

2. Look at your work for **1e** on page 369. Does the pair (0,4) represent the vector for a trip of 4 miles north? Does the pair (3,0) represent the vector for a trip of 3 miles east? What number pair represents the vector for the combination (sum) of the two trips? Inspect the following equation:





$$(0,4) + (3,0) = (3,4)$$

What is the distance between (3,4) and the origin? Does this correspond to your answer for **1e**?

3. Each of the following is a true equation. Can you determine how the addition is done?

a. $(1,1) + (2,2) = (3,3)$
b. $(1,3) + (4,3) = (5,6)$
c. $(2,0) + (-2,5\frac{1}{2}) = (0,5\frac{1}{2})$
d. $(-3,7) + (-2,-8) = (-5,-1)$
e. $(7,0) + (-2,4) = (5,4)$
f. $(a,b) + (c,d) = (a + c, b + d)$

4. In the process of adding vector number pairs, each number of one pair is added to the corresponding number of the other pair, and the sum is a number pair.

a. $(0,1) + (0,2) = ?$
b. $(2,2) + (1,-1) = ?$
c. $(7,-2) + (1,5) = ?$
d. $(6,3) + (-8,0) = ?$
e. $(-2,-1\frac{1}{2}) + (3\frac{1}{4},\frac{1}{2}) = ?$

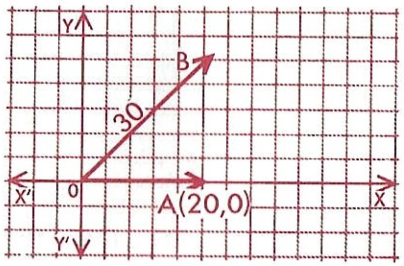
5. Does the addition of vectors (number pairs) exhibit closure? Is this addition commutative?

More about Vector Addition

The single vector which represents the sum of two (or more) vectors is called the **resultant.** The process of adding vectors to obtain a resultant is called **composition.**

As you have seen, when the vectors are given as number pairs, it is easy to find the sum (the resultant). Often, however, in actual situations, the data is not given in that convenient form.

1. Suppose you are asked to find the resultant of these forces:
Force A: A force of 20 grams acting to the east
Force B: A force of 30 grams acting to the northeast

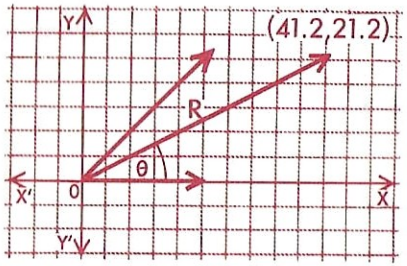


The diagram is shown at the left. To find the sum, we need the coordinates of the head of vector B.

Does $\cos 45° = \frac{x}{30}$? Find x.

Does $\sin 45° = \frac{y}{30}$? Name the coordinates of the head of vector B.

We now have (20,0) and (21.2, 21.2) as the heads of our two original vectors. Then is the head of the resultant given by the ordered pair (41.2,21.2)?



To find the magnitude of the resultant, we use the distance formula.

$$R = \sqrt{(41.2)^2 + (21.2)^2}$$
$$= 46 \text{ (approximately)}$$

This means we have a resultant force of approximately 46 grams.

2. Using one of the trigonometric functions, find the measure of $\angle\theta$. Describe the direction in which the force is acting.

EXERCISES

Add the forces. (Find the resultant: force and direction.)

1. A force of 30 pounds acting to the north
A force of 40 pounds acting to the northeast

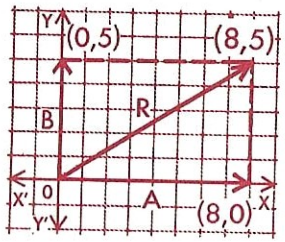
2. A force of 7 pounds acting to the east
A force of 4 pounds acting to the northeast

3. An airplane flies at a heading of N30°E at an air speed of 200 miles per hour. A wind of 40 miles per hour is blowing eastward. Find the resultant (speed and direction).

4. An airplane is flying at a heading of N65°E at 150 miles per hour. A wind of 30 miles per hour is blowing eastward. Find the resultant (speed and direction).

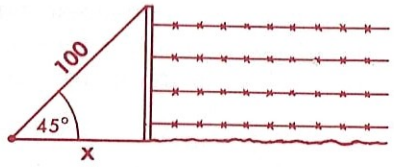
Vector Components

1. Study this diagram. Can vector R be described as the resultant of vectors A and B? We say also that vectors A and B are the **components** of vector R. Since A and B lie along the x-axis and y-axis, respectively, they are called the horizontal and vertical components of R.



Sometimes we are given a particular vector and are interested in finding only one component of this vector.

2. A guy wire is attached to a fence post and anchored to the ground such that it makes an angle of 45° with the horizontal; that is, with the ground. If the tension on the guy wire is 100 pounds, what is the horizontal tension on the fence?



3. A boy pushes on a lawnmower with a force of 30 pounds. If the handle makes an angle of 60° with the ground, how much force is effectively pushing the lawnmower along the ground? If the handle is lowered to make a 30° angle, what is the effective forward push? (Have you ever pushed a wheelbarrow? Did you lower the handles when you came to a bump or an incline?)

EXERCISES

Find the horizontal component of each of the following.

1. A force of 30 grams at an angle of 45° to the horizontal

2. A force of 60 grams at an angle of 30° to the horizontal

Find both the horizontal and the vertical components of each of the following.

3. A velocity of 250 miles per hour at an angle of 3° to the horizontal

4. A velocity of 175 miles per hour at an angle of 67° to the horizontal

Extra Practice Page 409

Challenge Page 424

1. When an angle in the coordinate plane has its vertex at the origin and one of its sides lies along the positive x-axis, the angle is said to be in ? ?. 351

2. Express, to the nearest hundredth, the tangent of an angle in standard position if the terminal side of the angle passes through (7,3). 352–354

3. One of the acute angles of a right triangle has a measure of 23°, and the leg adjacent to the angle is 18 feet long. What is the length of the leg opposite the angle? 354–356

4. Find, to the nearest degree, the measure of the angle described in **2.** 356–357

5. Express, to the nearest hundredth, the sine of an angle in standard position if the terminal side of the angle passes through (8,15). Find, to the nearest degree, the measure of the angle. 358–361

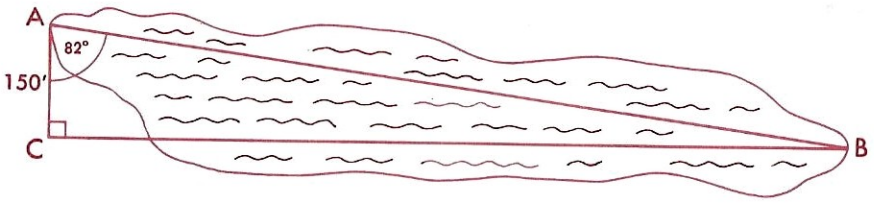
6. A straight stretch of hill has an angle of inclination of 6°. What is the rise in elevation per 500 feet of hill? 362–363

7. What is the cosine of the angle described in **5**? 365–366

8. One angle of a right triangle measures 46°, and the hypotenuse is 18.6 yards long. What is the length of the leg adjacent to the angle? 366

9. A ship sails N15°E for 150 miles. How far eastward has it gone? 368

10. To calculate the distance across a lake, a surveyor made the measurements as shown in the diagram. What is the distance across the lake? 368



11. Find the sum (resultant) of these pairs of vectors. 369–373

a. (2,3) and (4,1) **b.** (−3,5) and (5,3)

c. A force of 30 pounds acting toward the north and a force of 15 pounds acting toward the northeast

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

Elements of Probability 12

In his forecast a weatherman said that there was a chance of rain. What does a "chance" mean?

In your daily life there are many occasions when you must make a decision based on an estimate of the chance, or probability, that an event will occur. In this chapter we will study the mathematics of measuring a probability.

INTRODUCTORY IDEAS

Measuring Probabilities

1. Suppose a box contains ten marbles, five yellow and five red. You are asked to reach into the box and select a yellow marble, sight unseen. How many choices are there? How many of the choices are yellow? Thus, 5 of the 10 choices will result in a successful outcome. We say that the probability that a yellow marble will be selected is $\frac{5}{10}$, or $\frac{1}{2}$.

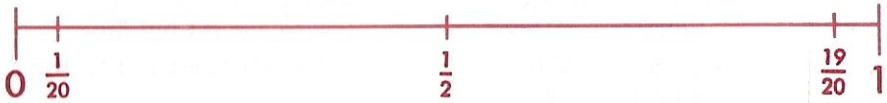
2. Suppose the box has fifteen marbles—ten red and five yellow. Again you are asked to select a yellow marble, sight unseen. How many possible choices are there altogether? How many of the choices are yellow? The probability that you will select a yellow marble is $\frac{5}{15}$, or $\frac{1}{3}$.

Imagine that you had been asked to select a red marble. What is the probability of selecting a red marble if the box has ten red marbles and five yellow marbles?

3. Next, consider a box that contains yellow marbles only—for instance, ten yellow ones and no red ones. You are to choose a marble, sight unseen. What is the probability that it will be a red marble? Since none of the possible choices is red, it is impossible that you will pick a red marble. The probability of a successful outcome is $\frac{0}{10}$, or 0.

In the same situation, what is the probability that the marble you select will be yellow? Since all ten of the marbles are yellow, the probability that a yellow marble will be selected is $\frac{10}{10}$, or 1.

4. You have seen that the measure of a probability can be a number from 0 to 1, inclusive. A number scale from 0 to 1 is shown below. Which point in this interval corresponds to the measure of a probability indicated by the phrase "a fifty-fifty chance"? "a certainty"? "an impossibility"? Which point might be said to correspond to "almost certain"? "almost impossible"?



The theory of probability has important applications in such widely different fields as statistics, economics, science, insurance, engineering, military planning, and medicine. However, because these applications are often quite complex, we will build our study using simple models of situations involving probability.

5. A box of marbles offered one such model. For another model, let us use a pencil whose cross section is hexagonal. Number the faces with the numerals 1, 2, 3, 4, 5, 6. Now roll the pencil on a flat surface. When it stops, one of the numerals will appear on the top face.

How many possible outcomes are there? Let us assume that each of these outcomes is equally likely.

a. How many of the numerals name even numbers? On any given roll of the pencil, what is the probability that an "even number" will occur?

b. What is the probability that a "5" will occur?

6. In answering **5a**, did you make use of the fact that there were six *possible outcomes*? You also made use of the fact that three of these outcomes would satisfy the required condition—namely, that an "even number" appear on the top face. Each such outcome is called a **favorable outcome** or a **success.**

In answering **5b**, did you also make use of the number of favorable outcomes and the number of possible outcomes? How many favorable outcomes are there for **b**? How many possible outcomes?

Let us use n to represent the number of possible outcomes in a given situation and f the number of favorable outcomes (outcomes that satisfy a required condition). Then do you see that, in each case, to find the measure of probability you found the quotient $\frac{f}{n}$?

▶ Measure of probability $= \frac{\text{number of favorable outcomes}}{\text{number of possible outcomes}} = \frac{f}{n}$

7. How many ways can a tossed coin land? (Exclude the rare possibility that it will stand on edge.) For a "normal" or "fair" coin, the outcomes of *heads* and *tails* are equally likely. What is the probability of the coin landing *heads*? What is the probability of the coin landing *tails*?

If, after tossing a coin a large number of times—1000 times, for instance—we found that the coin showed *tails* on $\frac{2}{3}$ of the tosses and *heads* on $\frac{1}{3}$ of the tosses, we would say that the coin was *not* a "fair" coin.

EXERCISES

1. As an experiment for you and your classmates, perform the following. Each of you toss a coin 20 times and record the number of heads and tails. Collect your results. If there are 30 students in your class, how many tosses will there be? Approximately how many of these tosses would you expect to result in heads? approximately how many in tails?

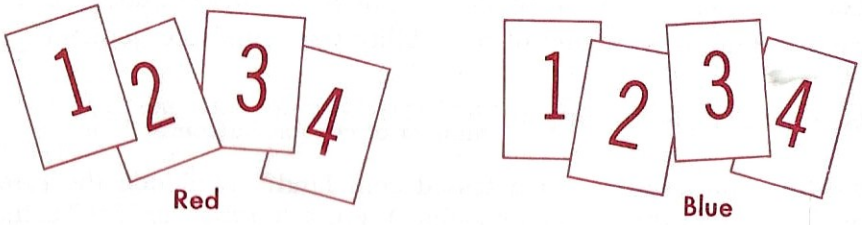
Find the actual result.' What is the per cent of difference between the expected result and the actual result?

2. Using the situation described in **5**, page 376 (the pencil with the hexagonal cross section), find the probability of each of the following.

a. An "odd number" will occur.
b. A "6" will occur.
c. The number named will be less than 1.
d. The number named will be less than 7.

3. Which of the events in Exercise **2** can be described as a certainty? an impossibility? a fifty-fifty chance?

Sample Spaces



1. Suppose we have two sets of four cards, one set red, the other blue, and the faces in each set are numbered as shown. Imagine that the cards are laid face down and you are asked to pick a card from each set. Thus, you might choose a red "4" and a blue "2." Or you might choose a red "1" and a blue "3." We can indicate these possibilities as (4,2) and (1,3) if we agree that the first number named in each pair represents the red card and the second number represents the blue card. How would you represent the choice of a red "3" and a blue "1"? the choice of a red "2" and a blue "3"?

2. Let us investigate how many ordered pairs are possible in this situation. With a red "1," can we have four different pairs: (1,1), (1,2), (1,3), and (1,4)? What pairs are possible with a red "2"? with a red "3"? with a red "4"?

3. Using the system at the right, we can show all the possibilities. Each ordered pair corresponds to a possible choice, and the first number in each pair is the number named on the red card. How many possible outcomes are there? Is this consistent with your answers in **2**?

Blue					
	4	(1,4)	(2,4)	(3,4)	(4,4)
	3	(1,3)	(2,3)	(3,3)	(4,3)
	2	(1,2)	(2,2)	(3,2)	(4,2)
	1	(1,1)	(2,1)	(3,1)	(4,1)
		1	2	3	4
		Red			

A systematic listing of all possible outcomes is called a **sample space**, or a **universal set.** Each outcome is called an **element** of the sample space, or a **sample point**, or, more briefly, a **point.**

4. Let us agree that any choice of a red card and a blue card is as likely as any other. You know that the number of possible outcomes (the number of points in the sample space) is 16.

Is the probability that (4,2) will be selected $\frac{1}{16}$? What is the probability that (1,3) will be selected?

5. Suppose we are interested in the probability that in a selection of a pair of cards the same number will be named on both cards. Will this condition be satisfied if the pair (1,1) is selected? if (2,2) is selected? (3,3)? (4,4)? Are there any other pairs that will satisfy the required condition?

Then there are 4 successful outcomes and 16 possible outcomes (equally likely). What is the probability that the same number will be named on both cards?

6. If a dime and a penny are tossed, the possible outcomes (heads or tails) can be recorded as follows:

$$(H,H) \quad (H,T) \quad (T,H) \quad (T,T)$$

These ordered pairs are the points in the sample space. The first entry in each pair tells whether a head or tail showed on the dime; the second entry refers to the penny. What is the number of possible outcomes? Is this the number of points in the sample space? How many of these points show one head and one tail? Then what is the probability that in tossing the two coins one will show a head and the other a tail?

EXERCISES

1. Suppose you have two sets of cards, one set red and the other blue, and the faces in each set are numbered 1, 2, 3, 4, 5, and 6. Imagine that the cards are laid face down and you are asked to pick a card from each set. Make a sample space. Indicate each of the possible outcomes with an ordered pair such as (2,5), where the first number in each pair indicates the number named on the red card and the second number in each pair indicates the number named on the blue card.

How many possible outcomes are there? (That is, how many points are there in the sample space?)

Retain the sample space for later use.

2. Suppose you are making a survey of families in which there are two children. You are interested in the number of boys and girls in these families. One of the possible outcomes of each interview may be recorded as (G,B). This ordered pair means that the older child is a girl and the younger is a boy. We can use such pairs as points in a sample space. List the other ordered pairs that are points in the sample space. How many points are there in the sample space?

3. Although it is not strictly true, assume that a child is as likely to be a boy as a girl. Find the probability of each of the following outcomes.

a. The older child is a boy and the younger is a girl.
b. Both children are boys.
c. Both children are of the same sex.

Truth Set of an Event

As on page 378, let us use the situation involving two sets of four cards, one set red and the other blue, with the faces in each set numbered 1, 2, 3, and 4. The sample space is shown here. Let us now consider the probability that the sum of the two numbers will equal 6. The points that satisfy this requirement are: (2,4), (3,3), (4,2). These points form a subset of the sample space. We call such a subset the *truth set* of the event. For convenience, a loop has been drawn around the points in this truth set.

Blue \ Red	1	2	3	4
4	(1,4)	(2,4)	(3,4)	(4,4)
3	(1,3)	(2,3)	(3,3)	(4,3)
2	(1,2)	(2,2)	(3,2)	(4,2)
1	(1,1)	(2,1)	(3,1)	(4,1)

We have assumed that the 16 points are equally likely to occur, so in this case the probability measure assigned to each point is $\frac{1}{16}$. It is possible that different probability measures may be assigned to the points in a sample space, but in any case the sun of those probability measures must be 1. Study this definition:

▶ The probability of an event is the sum of the probability measures of the points in its truth set.

Then the probability that the sum of the two numbers will equal 6 is: $\frac{1}{16} + \frac{1}{16} + \frac{1}{16}$, or $\frac{3}{16}$.

1. Are the results we obtained using the definition on page 380 consistent with the results you would have obtained using the definition involving $\frac{f}{n}$ (page 377)? Check and see.

2. Look back at your work for **5** on page 379. There, using the definition involving $\frac{f}{n}$, you found that the probability of the same number being named on both cards was $\frac{4}{16}$, or $\frac{1}{4}$. Do you obtain the same results using the definition on page 380?

The definition based on the "sum of the probability measures" can be applied more widely, since it does not require that the points in the sample space have equal probabilities.

3. As on page 376, imagine that there is a pencil having a hexagonal cross section, with its faces numbered from 1 to 6. After it is rolled, the pencil will come to rest with one of the numerals appearing on the top face. However, in this case, let us assume that the pencil is "biased"—for instance, it is not as likely that a "2" will appear as a "1," or not as likely that a "6" will appear as a "3." Let us further imagine that after considerable experience in rolling the pencil the following probabilities are assigned.

Occurrence	"1"	"2"	"3"	"4"	"5"	"6"
Probability	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

What should be the sum of these probabilities? Is it the sum?

4. Using the information in **3**, tell what the probability is that a number less than 4 will be named on the top face.

EXERCISES

1. Using the sample space you constructed for the sets of red and blue cards numbered from 1 to 6 (Exercise **1**, page 379), find the probability of each of the following.

a. The sum of the numbers will equal 7.
b. The sum of the numbers will equal 9.
c. The sum of the numbers will equal 11.
d. The sum of the numbers will equal 2.
e. The sum of the numbers will equal 1.

2. Using the "biased" pencil and the probabilities described in **3** above, find the probability of each of the following.

a. A number less than 5 will be named on the top face.
b. A number less than 7 will be named on the top face.

3. Three coins—a penny, a nickel, and a dime—are tossed together, each falling heads or tails with equal likelihood. We can construct a sample space as shown at the right. Each point is an ordered triple such as (H,T,T). This particular triple means that the penny shows heads, the nickel shows tails, and the dime shows tails. How many points are there in the sample space? Since there are 8 possible outcomes, what probability measure would you assign to each point? Find the probability of the following events.

P	N	D	
H	H	H	(H,H,H)
H	H	T	(H,H,T)
H	T	H	(H,T,H)
H	T	T	(H,T,T)
T	H	H	(T,H,H)
T	H	T	(T,H,T)
T	T	H	(T,T,H)
T	T	T	(T,T,T)

a. The three coins show heads.
b. There are two heads and one tail.
c. There are more tails than heads.
d. All coins fall the same way.

4. Suppose you are making a survey of families in which there are three children. One possible outcome in an interview with such a family is (B,G,B), which says that the oldest child is a boy, the second is a girl, and the youngest is a boy. Use a procedure similar to the one in Exercise **3** and construct the sample space of all possible outcomes. Are there 8 points in this space? List the members of the truth set of each of the following events, and find the probability of the event. (Assume that a child is as likely to be a boy as a girl.)

a. The family has two girls and one boy.
b. The family has more boys than girls.
c. The family has only boys.
d. The family has just one boy.
e. The oldest child is a boy.

Probability Trees

You know that in finding the probability of an event you need to know all the possible outcomes (that is, the sample space) and the truth set of the event. Sometimes it is easier to derive the sample space and the truth set by a device called a **probability tree.**

Let us consider the possibilities for numbers of boys and girls in families with three children. The first child may be either a boy or a girl. This is shown in Figure 1.

Now look at Figure 2. In a family having a boy as the oldest child, the next child can be either a boy or a girl. Similarly, in a family with a girl as the oldest child, the next child can be either a boy or a girl. Do you see that this is shown in Figure 2?

Figure 3 shows the possible outcomes for families with three children.

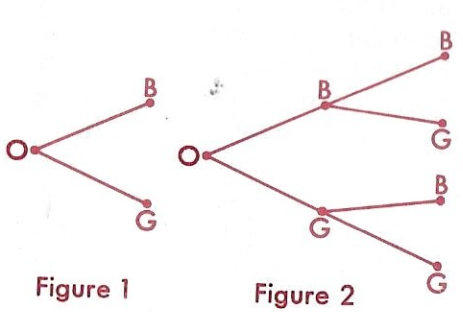


Figure 1

Figure 2

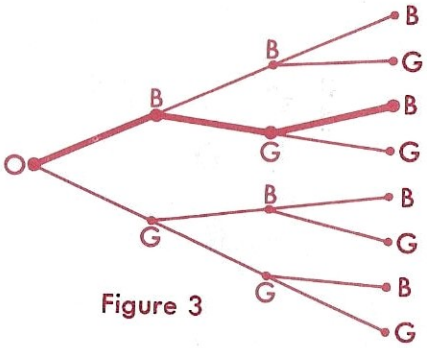


Figure 3

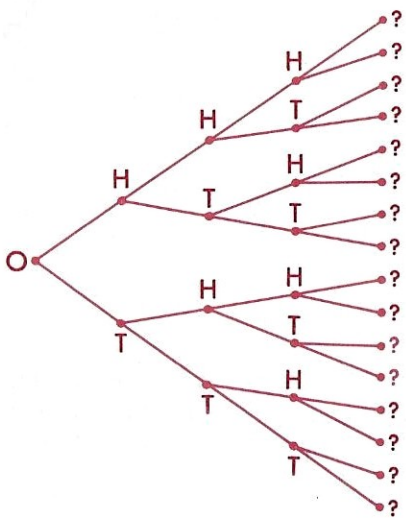
1. Start at O and follow the path BGB. It is marked in a heavier line. This path corresponds to the point (B,G,B) of the sample space. Follow the path marked BGG. Interpret it as a point in the sample space.

2. Starting at O, how many different paths can you trace? Does each path correspond to a different ordered triple? List these triples. Do you obtain the same set of triples that you obtained in Exercise **4**, page 382? Do you see that the 8 paths correspond to the 8 points in the sample space of families with three children?

3. Find the truth set of the event "The first and second children are boys." Does the truth set consist of the two paths BBB, BBG—in other words, the triples (B,B,B) and (B,B,G)? What is the probability that the first and second children are boys?

4. Find the truth set of the event "Only one child is a girl." How many ordered triples are in the truth set? What is the probability that only one child is a girl?

5. A coin is tossed four times. On each toss, it can show heads or tails with equal likelihood. To indicate the outcomes, we might adapt the system we have been using with pairs and triples and write, for instance, (H,T,T,T) to mean that a head showed on the first toss and a tail on each of the next three tosses. Without making a probability tree, try to make a listing of all the points in the sample space; that is, all the possible outcomes of the four tosses. Do you see that such a procedure can be somewhat difficult?



6. At the left is an uncompleted probability tree for the four tosses. Notice that with a probability tree the procedure is simple and direct. Copy and complete the tree.

7. How many paths are there? Is this the number of points in the sample space? If you have not already done so, list all the possible outcomes, using the form (H,T,T,T).

8. What is the probability that the four tosses of the coin will result in 4 heads? What is the probability that fewer than 4 heads will be the result?

EXERCISES

1. Using the probability tree you drew for **6** above, find the probability of each of the following.

- **a.** Two of the tosses will result in heads.
- **b.** Only one of the tosses will result in heads.
- **c.** None of the tosses will result in heads.

2. Two teams, A and B, play a match of three games. Team A can win (W) or lose (L) with equal likelihood. Make a tree showing the sample space if three games are played.

3. Using the probability tree you drew for Exercise **2**, find the probability of each of the following.

- **a.** Team A wins the first game.
- **b.** Team A wins the first two games.
- **c.** Team A wins at least one game.
- **d.** Team A wins at least two games.
- **e.** Team A wins exactly two games.
- **f.** Team A wins all three games.
- **g.** Team B loses all three games.

4. Draw a probability tree showing the possibilities for numbers of boys and girls in families with four children.

5. Using the probability tree you made for Exercise **4**, find the probability of each of the following.

- **a.** A family has four boys.
- **b.** All four children are of the same sex.
- **c.** A family has three girls.
- **d.** A family has at least three girls.

6. A doctor wants to test a new drug. He knows nothing about the drug except that it is harmless, and he starts with the assumption that the drug is as likely to cure (C) as to fail to cure (F). He tries the drug on five subjects. Make a tree showing the possible outcomes. Does your tree show 32 possible outcomes?

7. Using the probability tree you made for Exercise **6**, find the probability of each of the following.

- **a.** All five subjects are cured.
- **b.** Exactly four are cured.
- **c.** Exactly three are cured.
- **d.** Exactly two are cured.
- **e.** Exactly one is cured.
- **f.** No subjects are cured.

8. What should be the sum of the probabilities you found in Exercise **7a–f**, inclusive? Is it the sum?

EXTENDING THE USE OF PROBABILITY

Probability Symbol

1. Suppose a bag contains two green marbles and three yellow marbles, identical except for color. We can represent the two green marbles as g_1 and g_2; the three yellow marbles as y_1, y_2, and y_3. Suppose one marble is selected from the bag, sight unseen. Let us represent the outcome where g_1 is obtained as g_1; the outcome of g_2 as g_2; and so on. Then the sample space of choosing one marble is the set:

$$\{g_1,\ g_2,\ y_1,\ y_2,\ y_3\}$$

What is the number of possible outcomes? How many of these consist of choosing a green marble? Then the probability of choosing a green marble is ___?___.

Let g represent the choice of any green marble, and let y represent the choice of any yellow marble. For convenience, instead of saying "the probability of choosing any green marble," we can make use of a special probability symbol and write: $P(g)$. Similarly, $P(y)$ means "the probability of choosing any yellow marble."

What does $P(g)$ equal? $P(y)$? $P(g) + P(y)$?

2. Below is a sample space for the situation involving two sets of cards, one set red and the other blue, each set having six cards numbered 1 to 6. (Exercise **1**, page 379)

Blue \ Red	1	2	3	4	5	6
6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

You remember that the situation involves selecting one red card and one blue card. Let us use the variable r to represent the numbers named on the red cards and the variable b to represent the numbers named on the blue cards. Then we may interpret $P(r + b = 5)$ as "the probability that the sum of the numbers named on the red and blue cards is 5."

To find $P(r + b = 5)$, we can use the sample space. Is the pair (1,4) a member of the truth set? Is (2,3)? (3,2)? (4,1)? Are any other points members of the truth set? Then does $P(r + b = 5)$ equal $\frac{1}{36} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}$; that is, $\frac{1}{9}$?

Using the sample space you drew for the two sets of six cards, find each of the following.

	a	b	c
3.	$P(r = b)$	$P(r = 2)$	$P(b = 3)$
4.	$P(r + b = 1)$	$P(r + b = 7)$	$P(r + b > 6)$
5.	$P(r + b \leq 6)$	$P(r > b + 1)$	$P(r < b + 1)$

6. Find $P(r + b = 3) + P(r + b < 3) + P(r + b > 3)$.

7. Find mentally: $P(r = 5) + P(r \neq 5)$.

Probability of a Conjunction

Let us again make use of the two sets of cards numbered from 1 to 6 and the sample space.

1. As before, r represents the numbers named on the red cards; b represents the numbers named on the blue cards. List all the points that are in the truth set of the event $r = 1$. Are these the points that are enclosed in the solid loop, or are they the points enclosed in the dashed loop?

List all the points that are in the truth set of the event $b = 2$.

Do the two sets of points have any point in common? Name it.

Blue \ Red	1	2	3	4	5	6
6	(1,6)	(2,6)	(3,6)	(4,6)	(5,6)	(6,6)
5	(1,5)	(2,5)	(3,5)	(4,5)	(5,5)	(6,5)
4	(1,4)	(2,4)	(3,4)	(4,4)	(5,4)	(6,4)
3	(1,3)	(2,3)	(3,3)	(4,3)	(5,3)	(6,3)
2	(1,2)	(2,2)	(3,2)	(4,2)	(5,2)	(6,2)
1	(1,1)	(2,1)	(3,1)	(4,1)	(5,1)	(6,1)

As you know, the subset common to two sets is called the *intersection* of the two sets. Is (1,2) the intersection of the sets {(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)} and {(1,2), (2,2), (3,2), (4,2), (5,2), (6,2)}?

2. Suppose you are asked to determine the probability that the selection of a pair of cards will result in a red "1" and a blue "2." We can represent this as $P(r = 1 \text{ and } b = 2)$. You can see that the event $r = 1$ *and* $b = 2$ has the form of a conjunction. In your work with open sentences in earlier chapters, you found that the truth set (solution set) of a conjunction is the intersection of the truth sets of the two parts of the sentence. Is (1,2) the only point for which it is true that r equals 1 *and* b equals 2?

Then does $P(r = 1 \text{ and } b = 2)$ equal $\frac{1}{36}$?

3. List the truth set of $r + b = 5$. List the truth set of $r > b$. Are there any points common to both sets? How many? Name them. Does $P(r + b = 5 \text{ and } r > b)$ equal $\frac{1}{36} + \frac{1}{36}$; that is, $\frac{1}{18}$?

▶ To find the probability of the conjunction of two events, find the probability of the intersection of the truth sets of these events.

EXERCISES

For each of Exercises **1–7**, use the sample space for the two sets of cards numbered from 1 to 6. List the points in the truth set of each part of the conjunction; then list the point or points of intersection of these truth sets. Then find the probability of the conjunction.

1. $P(r = b \text{ and } r > 3)$
2. $P(r < b \text{ and } b = 2)$
3. $P(r = b + 1 \text{ and } r = 4)$
4. $P(r = b \text{ and } r < 4)$
5. $P(r > b \text{ and } r + b = 7)$
6. $P(b = 2r \text{ and } r < 6)$

7. Find $P(r = 2b \text{ and } b = 2r)$. How many points are common to the truth set of $r = 2b$ and the truth set of $b = 2r$? Then is their intersection the null set? Is the event $r = 2b$ *and* $b = 2r$ a certainty, or is it an impossibility? Then what is the probability of this event?

8. Using a probability tree, construct a sample space showing all possible outcomes when a penny, a dime, and a quarter are tossed and allowed to fall heads (H) or tails (T). One of the ordered triplets in this space is (T,H,T), which means that the penny falls tails, the dime falls heads, and the quarter falls tails. Does your sample space contain 8 points? Find the probability of the following conjunctions.

a. Two coins fall heads *and* the quarter falls heads.
b. Two coins fall the same way *and* the dime falls tails.
c. All three coins fall the same way *and* the dime and quarter fall tails.
d. All three coins fall the same way *and* the dime falls differently.

9. Set up a sample space to show all the possible outcomes when one card is chosen from each of these two sets of cards: The first set consists of three white cards numbered 3, 4, and 5. The second set consists of four black cards numbered 1, 2, 3, and 4. How many possible outcomes are there? What is the probability associated with each point? Let w represent the numbers on the white cards and b the numbers on the black cards. Find the following probabilities.

a. $P(w = 3 \text{ and } b = 4)$
b. $P(w \neq 3 \text{ and } b = 4)$
c. $P(w > 4 \text{ and } b < 2)$
d. $P(w < 4 \text{ and } b < 2)$
e. $P(w > 4 \text{ and } b > 2)$
f. $P(w < 4 \text{ and } b > 2)$

Probability of a Disjunction

Two equally matched bowling teams, A and B, have a three-game match. The triple (A, A, B) represents one of the possible outcomes, all of which are equally likely. This particular triple shows that team A won the first two games and B won the third game.

1. Set up a sample space of the possible outcomes in the match.

2. The truth sets of two events are listed below.

A wins at least two games	B wins the first game
(A,A,A)	(B,B,B)
(A,A,B)	(B,B,A)
(A,B,A)	(B,A,B)
(B,A,A)	(B,A,A)

Let us now consider the event "A wins at least two games *or* B wins the first game." As you know, this is a disjunction, and it is true if either (or both) of its parts is true. List the truth set of this disjunction. Does it contain 8 distinct points? 7 distinct points? We cannot list (B,A,A) twice as a member of the truth set of the disjunction since we would be listing the same outcome twice.

Therefore, the truth set of the disjunction consists of the 7 points listed at the right.

(A,A,A)	(B,B,B)
(A,A,B)	(B,B,A)
(A,B,A)	(B,A,B)
(B,A,A)	

3. Study the work in **2** and find what the probability is that:

a. A wins at least two games.
b. B wins the first game.
c. A wins at least two games *or* B wins the first game.

As you know, the set containing all the elements that belong to either (or both) of two sets is called the *union* of these sets.

▶ To find the probability of the disjunction of two events, find the probability of the union of their truth sets.

4. a. List the members of the truth set of the event "A wins the first game."

b. List the members of the truth set of the event "B wins the second game."

c. List the members of the truth set of the disjunction "A wins the first game *or* B wins the second game." Are there 8 members in the truth set? 7? 6? 5?

If you decided that the truth set has 8, 7, or 5 members, recheck your listing. Remember that we do not list the same outcome more than once as a member of the union of two sets.

5. What is the truth set of the event "A wins all three games"? What is the truth set of the event "B wins all three games"? What is the truth set of the disjunction "A wins all three games *or* B wins all three games"? What is the probability of this disjunction?

6. When two sets have no members in common, they are called **disjoint sets.** Look back over your work in **5.** Does the truth set of the event "A wins all three games" have any members in common with the truth set of the event "B wins all three games"? Are these, then, disjoint sets?

Is the number of members in the union of two disjoint sets simply the sum of the number of members in each set?

7. List the members of the truth set of the event "A wins exactly two games." List the members of the truth set of "B wins only the first and third games." Are these disjoint sets? How many members are in the union of these two sets?

Is this union the truth set of the disjunction "A wins exactly two games or B wins only the first and third games"?

What is the probability of this disjunction?

EXERCISES

1. Suppose you are making a survey of political opinion. You may interview a Republican (R), a Democrat (D), or an Independent (I). Set up the sample space of all possible combinations of interviews with two people. One point in this space will be (R,I). It represents a pair of interviews, the first with a Republican and the second with an Independent. Does your sample space include 9 points?

2. List the truth set of the following disjunctions.
a. Exactly one Republican is interviewed *or* exactly one Democrat is interviewed.
b. Exactly two Republicans are interviewed *or* exactly two Democrats are interviewed.
c. At least one Democrat is interviewed *or* at least one Republican is interviewed.

3. Assume that interviews with a Republican, a Democrat, and an Independent are equally likely. Find the probability of each disjunction in Exercise **2.**

4. An examination consists, in part, of four True-False items. Assume that a student knows nothing about these items and guesses the answers. He is as likely to guess T as F. One point in the sample space is (T,T,F,T). Explain what this point indicates. Set up the sample space of all possible combinations of guesses. Are there 16 points in your sample space?

5. List the points in the truth set of each of these events. Notice that some are simple events and others involve conjunction and disjunction.

a. There are as many T as F answers.

b. There are more T than F answers.

c. The first three answers are the same *and* the last answer is different.

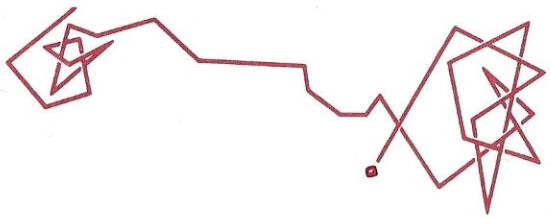
d. The first two answers are different *and* the last two answers are the same.

e. The first two answers are the same *and* the last two answers are the same.

f. All four answers are the same *or* three answers are the same.

6. Find the probability of each of the events in Exercise **5.**

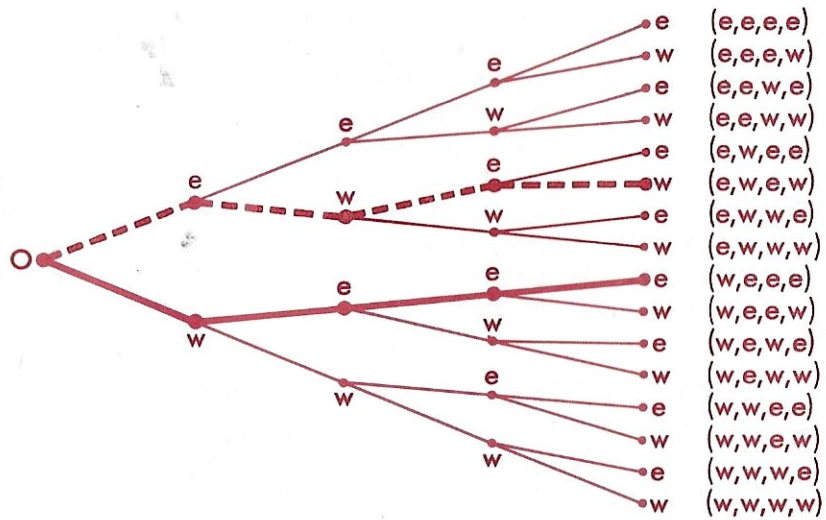
Probability in Science



The zig-zag path of a particle is called Brownian motion.

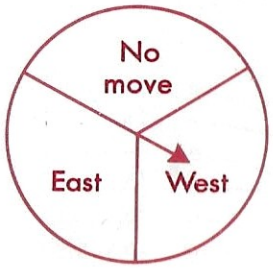
The motion of molecules is of considerable interest to physicists. The paths of molecules are determined in part by collisions with each other. Since the number of molecules is so large and there would be too many paths to trace, the theory of probability becomes useful. The theory applied to these motions is far too complex for this course, but we can discuss a simplified version.

Suppose an object can move east or west a distance of 1 inch each time it moves. A coin is tossed. If the coin falls heads, the object moves 1 inch to the east. If the coin falls tails, the object moves west. The coin is tossed four times. This probability tree and sample space shows all the possibilities.


O
e
w
e
w
e
w

3. What is the probability of each event in **2a–f**?

4. Which are the most likely final positions?



EXERCISES

1. This diagram shows a dial that we will use to determine whether an object will move 1 inch east, 1 inch west, or will not move. The spinner is just as likely to stop in one sector as another.

The spinner is to be spun three times. Using *e* to indicate that the object moves 1 inch east, *w* to indicate that the object moves 1 inch west, and *x* to indicate that the object does not move, construct a probability tree that shows all the possibilities. List all the possible outcomes as ordered triples. How many possible outcomes are there?

2. List the points in the truth set of each of these events.

a. The object is 3 inches east of the starting point.
b. The object is 2 inches west of the starting point.
c. The object is 1 inch from the starting point.
d. The object is at the starting point.

3. What is the probability of each event in Exercise **2a–d**?

4. Which final positions are most likely for the object in Exercise **1**?

1. The numbers used to measure probabilities are between ? and ?, inclusive. 375–376

Extra Practice Page 410

2. In an experiment, there are seven possible outcomes, each of which is as likely to happen as another. If three of these are favorable to a certain purpose, what is the probability of a favorable outcome? of an unfavorable outcome? 376–377

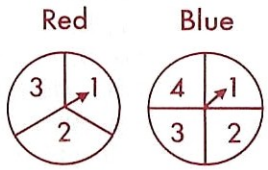
Challenge Page 425

3. Suppose we have two sets of three cards, one set red and the other white. The faces of the cards in each set are numbered 1, 2, and 3, respectively. The cards are laid face down, and one card is chosen from each set. One possible outcome can be represented by the ordered pair (1,2), which means that a red "1" and a white "2" are chosen. Draw a table showing the sample space. 378–380

4. Tell the meaning of each of the following terms: sample space; sample point; truth set of an event; probability of an event. 378–381

5. What must be true about the sum of the probability measures assigned to the points of a sample space? 380–381

6. A red disk is divided into three equal sectors numbered 1, 2, and 3. A blue disk is divided into four equal sectors numbered 1, 2, 3, and 4. By spinning the needle of each, a pair of numbers (r,b) is determined. Draw a table showing the sample space of outcomes and find the probabilities of the following events, assuming that the points in the sample space have equal probabilities. 383–384



a. The numbers named on the red and blue disks are equal.
b. The number r on the red disk is greater than the number b on the blue disk.
c. The sum of the numbers named is 7.

7. Referring to your work with the two disks in **6**, list the truth set of these events. 386–391

a. $r = 3$
b. $b = 4$
c. $r = 3$ *and* $b = 4$
d. $r = 3$ *or* $b = 4$

8. What is the probability of the events in **7a–d**? 386–391

If you had difficulty with any of the work on this test, find help on the pages listed at the ends of the rows.

Extra Practice

For use with pages 1–41

Find the replacement that makes each of the following sentences true. Then state whether each sentence illustrates: the commutative principle of addition (CA); associative principle of addition (AA); commutative principle of multiplication (CM); associative principle of multiplication (AM); distributive principle (D).

1. $5 \times (6 \times a) = (5 \times 6) \times 4$
2. $2\frac{1}{2} + b = 3\frac{1}{4} + 2\frac{1}{2}$
3. $5 \times (6 + c) = (5 \times 6) + (5 \times 3)$
4. $2\frac{1}{3} \times r = 1\frac{1}{2} \times 2\frac{1}{3}$
5. $(1\frac{1}{8} + d) + 4\frac{1}{4} = 1\frac{1}{8} + (1\frac{3}{4} + 4\frac{1}{4})$
6. $\frac{2}{3} + \frac{3}{5} = \frac{3}{5} + k$
7. $(3\frac{1}{2} + 2\frac{1}{2}) + g = 3\frac{1}{2} + (2\frac{1}{2} + 0)$
8. $1.1 \times 3.3 = s \times 1.1$

9. Complete the indicated operations in modulo 3.
   a. $2 \otimes (2 \oplus 2)$
   b. $[2 \otimes (1 \oplus 2)] \oplus [1 \otimes (2 \oplus 0)]$
   c. $1 \oplus 1 \oplus (2 \otimes 2)$
   d. $(1 \otimes 0) \oplus 1 \oplus (1 \otimes 1)$

10. Define sets *A*–*D* by description; sets *E*–*G* by tabulation.
   $A = \{1, 2, 3, \ldots, 8\}$
   $B = \{6, 8, 10, \ldots\}$
   $C = \{4, 8, 16, 32, \ldots\}$
   $D = \{.5, 5, 50, 500, \ldots\}$
   $E = \{\text{the odd counting numbers less than } 20\}$
   $F = \{\text{all even counting numbers less than } 15 \text{ and greater than } 2\}$
   $G = \{\text{all odd counting numbers between } 3 \text{ and } 5\}$

11. Write each of the following as a mathematical sentence.
   a. r is greater than 1.
   b. x is less than or equal to 4.
   c. y is not equal to 4.
   d. $2m$ is greater than or equal to 3.
   e. $4x$ is equal to 7.
   f. y is less than 1.

12. Find the solution set of each sentence you wrote in **11**. The replacement set for the variable is {0,1,2,3,4,5,6}.

13. On a line, draw a graph of the solution set of each of the sentences in **11** except **11e**.

14. Evaluate each of the following at 1; at 3; at 5.
   a. $\frac{x^4}{x^4}$
   b. $2 \cdot (x^2 + 3)$
   c. $\frac{x^3 + 9}{2}$
   d. $\frac{x^4}{x^3}$
   e. $\frac{x^6}{x^3}$
   f. $(3 + x) \cdot 5\frac{1}{2}$
   g. $\frac{6}{x^2}$
   h. $\frac{x + 7}{x + 2}$

15. Replacing a by 3, m by 4, and n by 2, evaluate:
   a. a^{m+n}
   b. a^{m-n}
   c. a^{m-2n}
   d. a^{2m-3n}

Find the replacement for the variable that makes each sentence true. The replacement set is the set of integers.

	a	b	c	d
1.	$x + (-6) = 0$	$2 + n = 2$	$9 + y = 0$	$-1 + s = 0$
2.	$r + 0 = -11$	$t + 1 = 0$	$x + x = 0$	$m = 1 + (-4 + 4)$

Add.

3.	$8 + (-2)$	$-3 + (-4)$	$-8 + 2$	$-15 + 15$
4.	$11 + (-3)$	$11 + (-5)$	$11 + (-7)$	$11 + (-12)$
5.	$4 + (-6)$	$4 + (-9)$	$4 + (-15)$	$4 + (-25)$
6.	$-4 + (-2)$	$-4 + (-8)$	$-4 + (-10)$	$-4 + 1$
7.	$15 + (-14)$	$15 + (-8)$	$15 + (-11)$	$15 + (-3)$

Replace each question mark with the integer that will make the sentence a true statement.

	a	b	c
8.	$\lvert 1 \rvert = ?$	$\lvert -1 \rvert = ?$	$\lvert -12 \rvert = ?$
9.	$\lvert 31 \rvert = ?$	$\lvert 0 \rvert = ?$	$\lvert 156 \rvert = ?$
10.	$\lvert -2 + 5 \rvert = ?$	$\lvert 2 + (-5) \rvert = ?$	$\lvert -2 + (-5) \rvert = ?$

Subtract.

11.	$2 - 6$	$2 - 5$	$2 - (-1)$
12.	$4 - 0$	$4 - (-2)$	$4 - (-4)$
13.	$6 - (-3)$	$6 - (-4)$	$6 - (-7)$
14.	$-1 - 0$	$-1 - 1$	$-1 - 3$
15.	$-2 - (-5)$	$-2 - (-3)$	$-2 - (-1)$

Multiply.

16.	$3 \cdot (-9)$	$-4 \cdot 12$	$-6 \cdot (-8)$
17.	$8 \cdot (-12)$	$-11 \cdot (-20)$	$-7 \cdot 9$
18.	$12 \cdot (-15)$	$-12 \cdot (-12)$	$14 \cdot 16$
19.	$[4 \cdot (-6)] \cdot 3$	$[-3 \cdot (-4)] \cdot 4$	$6 \cdot [4 \cdot (-3)]$
20.	$-5 \cdot [-5 \cdot (-5)]$	$6 \cdot (-7 \cdot 8)$	$3 \cdot [6 \cdot (-9)]$

Divide.

21.	$39 \div (-3)$	$-39 \div 13$	$-39 \div (-13)$
22.	$-96 \div 48$	$-48 \div (-48)$	$-102 \div (-17)$
23.	$-112 \div 16$	$42 \div (-14)$	$-108 \div (-54)$

Multiply. Simplify answers wherever possible.

	a	b	c	d
1.	$\frac{1}{-2} \cdot \frac{-1}{4}$	$\frac{-2}{-5} \cdot \frac{-5}{7}$	$\frac{15}{4} \cdot \frac{16}{-3}$	$\frac{3}{2} \cdot \frac{-2}{3}$
2.	$\frac{-3}{-8} \cdot \frac{-2}{9}$	$\frac{10}{-3} \cdot \frac{9}{-7}$	$\frac{-35}{5} \cdot \frac{-15}{-7}$	$\frac{5}{3} \cdot (-3)$
3.	$\frac{-4}{15} \cdot \frac{-5}{2}$	$\frac{-1}{3} \cdot \frac{9}{-4}$	$\frac{12}{3} \cdot \frac{-1}{6}$	$-7 \cdot \frac{33}{-14}$

4. What is the multiplicative inverse of -7? $\frac{1}{3}$? $\frac{-2}{3}$? $\frac{7}{2}$? $-1\frac{1}{3}$?

Divide. Simplify answers wherever possible.

5.	$\frac{1}{5} \div \frac{-2}{5}$	$-12 \div \frac{4}{5}$	$\frac{-4}{7} \div \frac{3}{7}$	$-8 \div (-9)$
6.	$\frac{-2}{-7} \div \frac{8}{-9}$	$\frac{-9}{11} \div \frac{-2}{-3}$	$\frac{-6}{13} \div \frac{-3}{13}$	$0 \div \frac{3}{5}$
7.	$-30 \div \frac{-3}{13}$	$\frac{15}{-16} \div \frac{3}{-4}$	$-20 \div \frac{-2}{5}$	$\frac{-5}{2} \div \frac{1}{6}$

Add.

8.	$\frac{-8}{9} + \frac{-1}{9}$	$\frac{4}{11} + \frac{-8}{11}$	$\frac{-3}{17} + \frac{-7}{17}$	$\frac{8}{-3} + \frac{-8}{-3}$
9.	$\frac{-1}{2} + \frac{-1}{4}$	$\frac{-4}{7} + \frac{5}{7}$	$\frac{-2}{3} + \frac{-4}{5}$	$\frac{5}{7} + \frac{-5}{14}$
10.	$\frac{-2}{3} + \frac{1}{4}$	$\frac{-7}{2} + \frac{-6}{7}$	$\frac{-7}{3} + \frac{-6}{15}$	$-1\frac{1}{2} + 2\frac{1}{2}$

11. What is the additive inverse of $\frac{-1}{2}$? of $\frac{3}{4}$? $\frac{5}{-6}$? $\frac{-2}{3}$?

Subtract. Simplify answers wherever possible.

	a	b	c	d
12.	$\frac{-2}{5} - \frac{4}{5}$	$\frac{-5}{17} - \frac{9}{17}$	$\frac{5}{13} - \frac{-7}{13}$	$\frac{-1}{3} - \frac{4}{3}$
13.	$\frac{4}{7} - \frac{6}{7}$	$\frac{1}{5} - \frac{-13}{15}$	$\frac{-13}{15} - \frac{-7}{15}$	$\frac{-8}{9} - \frac{-8}{9}$
14.	$\frac{11}{19} - \frac{-4}{5}$	$\frac{-3}{4} - \frac{3}{2}$	$\frac{7}{5} - \frac{-2}{11}$	$\frac{11}{6} - \frac{-1}{7}$

15. Using the symbol "<," order the numbers in each of the following.

$\frac{1}{10}, 0, \frac{1}{4}$ $\quad \frac{-13}{2}, \frac{-14}{2}, \frac{-15}{2}$ $\quad \frac{1}{10}, \frac{1}{9}, \frac{1}{8}$ $\quad \frac{-1}{10}, \frac{-1}{9}, \frac{-1}{8}$

16. Find a number that is between $\frac{8}{9}$ and $\frac{9}{10}$; $\frac{-9}{8}$ and $\frac{-10}{9}$.

Using the set $\{-3, -2, -1, 0, 1, 2, 3\}$ as the replacement set, find the solution set of each of the following open sentences.

	a	b	c
1.	$3 + x = 4$	$5 - y = 5$	$r > 0$
2.	$2y + 3 = 1$	$m + 1 = 0$	$n + 1 < 0$
3.	$x - 2 > 2$	$3 < y + 2$	$3y < -9$

The set of rational numbers is the replacement set for the remainder of the work on this page.

Which of the following sentences are equivalent to $3x - 8 = 5$?

a. $3x - 6 = 7$ b. $3x - 2 = 11$ c. $3x = 14$
d. $13 = 3x$ e. $x = \frac{1}{3} \cdot 13$ f. $4\frac{1}{3} = x$

5. Which of the following are equivalent to $5y + 7 < 3 + (-1)$?

a. $5y + 7 < 4$ b. $5y < 5$ c. $5y < -5$ d. $10y - 2 < -12$

Can the solution set of the original sentence be represented as $\{y \mid y < -1\}$?

Find the solution set of each of the following.

	a	b	c
6.	$3m = 15$	$-8 = 2x$	$y + 3 < 8$
7.	$x - 7 = 12$	$\frac{1}{4}r = -2$	$\frac{2}{3}x \geq 6$
8.	$3z \geq 5$	$6 = 1\frac{1}{2}y$	$m + (-2) = 17$
9.	$2x - 6 = 12$	$3y + 2 < 13$	$\frac{1}{3}r - 2 = 5$
10.	$\frac{3}{4}x + 12 = 18$	$5 \leq 1\frac{1}{2}m - 4$	$3r - 12 < -6$
11.	$6 - 2n = 8$	$3 - \frac{1}{3}x < 0$	$2y + 1.5 = -5.5$
12.	$3x - 2 = 5x + 2$	$t + 5 \geq 2t - 1$	$4y - 16 = y + 8$
13.	$6 - 2y > 9 - 3y$	$\frac{1}{3}x + 6 = \frac{2}{3}x - 9$	$\frac{1}{2}m + 5 < m - 2$
14.	$9 - 2x = 6 - 2x$	$\frac{3}{4}t + 27 < \frac{1}{4}t - 1$	$1\frac{1}{2}x + 2 = 5 - 2x$

On a number line, draw the graph of the solution set of each of the following.

15. a. $x < 3$ *and* $x < 8$ b. $x = \frac{-1}{4}$ *or* $x = \frac{1}{4}$
16. a. $x > -4$ *and* $x < 0$ b. $x \geq -1$ *or* $x = -2$

17. The sum of five consecutive odd integers is 75. What are the integers?

18. One sixth of the sum of three consecutive integers is 21. What are the integers?

1. Which of the following are polynomials?

a. 3 **b.** $2y$ **c.** $\frac{3}{x}$ **d.** $\frac{x}{4}+3$
e. $4x^2+3x-1$ **f.** $a^3-\frac{1}{a}$ **g.** $\frac{1}{2}a^2+6a-14$

2. Evaluate each of the polynomials in **1** that contain a variable at 1; at -1; at 0; at 3; at -3.

3. Add.

a. $2y+1$ / $2y-1$

b. $3x^2+2x$ / $-4x^2-3x$

c. a^2-3a-4 / $-a^2+3a-4$

d. $x^4-x^3-6x^2+4$ / x^3+8x^2

e. m^3+5 / $-\frac{1}{2}m^3$

f. x^3-8x^2+5 / $-x^3-8x^2-5$

g. $\frac{1}{6}y^3-2y+7$ / $\frac{4}{5}y^3+y$ / $\frac{1}{3}y^3+3y-6$

h. $\frac{1}{8}x^2+x-5$ / $\frac{1}{5}x^2-x-7$ / $\frac{3}{4}x^2+\frac{1}{2}x+9$

i. $.03n^2+.4n+.01$ / $.39n^2-.44$ / $.62n^2-.6n+.45$

4. Add each of the following pairs of polynomials.

a. $3-x^2$ and $x-x^2$ **b.** y^3-4y^2 and $4y^2+2$
c. a^4+a^2 and $4+a^2-a^3$ **d.** $2-3m^2$ and $3m^3-m^2+2$
e. $\frac{1}{2}+x^2$ and $\frac{1}{7}-x+x^2$ **f.** g^3-4+g and $5g$

5. Evaluate each sum in **4** at 2; at -2.

6. What is the additive inverse of each of these polynomials?

a. 0 **b.** x **c.** $1-x$ **d.** $1+x$
e. $1-x-x^2$ **f.** $1+x-x^2$ **g.** $-x^3$

7. Subtract.

a. $8x$ / $9x$

b. $3x^2-2$ / $2x^2+3$

c. y^3-y^2+y / y^3+y^2+y

d. $7x^3-2x^2-1$ / $-7x^3-2x^2+1$

e. $x^2-\frac{1}{2}x+1$ / $-2x^2-2x+3$

f. $\frac{4}{5}y^2-\frac{1}{2}y$ / $-\frac{3}{7}y^2-\frac{1}{4}y$

g. $-5x^3+5x^2-7-8x$ / $-5x^2-5x^3+x-9$

h. $\frac{2}{3}g^3-2g+\frac{1}{4}g^2-5$ / $-\frac{3}{5}g^3-\frac{1}{3}g^2+3g-4$

8. Perform the indicated operations.

a. $(3x^3+x^2-1)-(2x^3+2x^2+1)$
b. $(3y^3-y)-(y^3-y+1)$
c. $(x^3+3x-1)+(-x^2+2x+2)-(x^2+4x)$
d. $(6y^3-2y^2+4)+(-3y^3-4y^2+2)-(-2y^3-2y^2+6)$
e. $(7n^4+3n-2)+(8n^2+5n^4-6)-(5n^3+2n^2-3n^4+4)$

1. Express each as a power of the base.

a. $(r^3)^4$ **b.** $(x^4)^2$ **c.** $(n^0)^0$ **d.** $(e^0)^2$ **e.** $(y^3)^5$

2. Find the products.

a. $y(y-4)$ **b.** $2x(x+3)$ **c.** $3x(x^2-x-6)$
d. $r(r^3-r^2)$ **e.** $(g+5)g^2$ **f.** $-2x(x-2x^2+6)$
g. $a^2(a^2+a)$ **h.** $4p^2(3p^3+6p-1)$

3. Factor the following.

a. $3a^2-9$ **b.** $4x^2+4x$ **c.** $9b^2-36b$
d. $6x^2-14x-16$ **e.** y^3-y^2-y **f.** $5(x-y)+z(x-y)$
g. $16xy+8x+24xz$ **h.** $10xyz+5xyz^2$

Find the products.

	a	b	c
4.	$(x-3)(x-3)$	$(y+x)(-y+x)$	$(5x^2-7)(-2x^2+6)$
5.	$(2g+4)(g-5)$	$(y^2+2)(y^2+8)$	$(15r^3-4)(2r^3-5)$
6.	$(3x-6)(4x-5)$	$(x^2-9)(-x^2+6)$	$(8n^3-5)(4n^3-5)$
7.	$(4z-8)^2$	$(7r-1)^2$	$(\frac{1}{4}g^2+9)^2$
8.	$(y-3)^2$	$(2a+4)^2$	$(5y^2-2y)^2$
9.	$4x^2-3x+2$ / $-3x-3$	z^2+4z-3 / $z+2$	a^2+6a-4 / $2a^2-2a$

Solve.

	a	b
10.	$(x-8)^2=(x-9)(x-6)$	$(z+12)^2=(z+8)(z+20)$
11.	$(y-5)^2=(y+7)(y-9)$	$(r+4)^2=(r-4)^2+10r+12$
12.	$(b+1)^2=(b-5)(b+3)$	$(a-9)^2=(a+3)(a-13)$

Factor completely.

	a	b	c
13.	$n^2+12n+36$	x^2-4x+4	$z^2+14z+49$
14.	b^2-81	r^2-100	a^2-b^2
15.	$5x^2-125y^2$	$2y^2-18x^2$	n^4-n^8
16.	$25x^2-12x+36$	$81y^2+225y+100$	$.04x^2-.12x+.09$
17.	y^2+6y+8	$x^2+3x-54$	$z^2-2z-48$
18.	$6y^2-y-12$	$10x^2-29x+10$	$6n^2-7n-5$
19.	$4x^2-64$	$30y^2+60y+30$	$7s^2+35s-168$
20.	$2xyz-6xy$	$9a^2b^3-6a^4b^2$	$2a^2-7ab-15b^2$

Perform the operations indicated in each exercise. Reduce all answers to lowest terms.

	a	b
1.	$\frac{45x^2y}{7} \cdot \frac{1}{30xy^2}$	$\frac{3t}{14c^2} \cdot \frac{7ac}{6ct}$
2.	$\frac{x+2}{x^2-4} \cdot \frac{x^2-2x+1}{x-2}$	$\frac{z-4}{z^2-1} \cdot \frac{3z+3}{z^2-3z-4}$
3.	$\frac{x-2}{x+2} \cdot \frac{x+2}{x^2-4x+4}$	$\frac{3y-3}{y+2} \cdot \frac{2y+2}{y^2-1}$
4.	$\frac{2x}{3b} + \frac{3x^2}{ab}$	$\frac{2x}{3y} + \frac{a}{3x}$
5.	$\frac{3b}{2x} - \frac{9b}{5x}$	$\frac{2c}{4} - \frac{1}{c^2}$
6.	$\frac{3}{x+3} + \frac{x}{2x+6}$	$\frac{x^2-2x}{x^2-1} + \frac{x}{x+1}$
7.	$\frac{7x}{2x-2} - \frac{3}{x-1}$	$\frac{2-x}{x^2-4} - \frac{1}{x}$
8.	$x + \frac{3}{x+1}$	$3y - \frac{y}{2+y}$
9.	$\frac{\frac{3}{x}}{2-\frac{1}{x}}$	$\frac{\frac{3}{x}}{x}$
10.	$\frac{x-3}{2x+6} \div \frac{x-3}{4x-12}$	$\left(1+\frac{3}{x-2}\right)\left(2-\frac{6}{x+1}\right)$

Solve the following equations. Check your answers.

11.	$\frac{x}{2} + \frac{x}{3} = 5$	$\frac{1}{x} - \frac{2}{3x} = 1$
12.	$\frac{y+2}{2} = \frac{4y-7}{3}$	$\frac{x+2}{4} = 2 + \frac{x+3}{6}$
13.	$\frac{2x+1}{x-5} = \frac{8x-1}{4x+5}$	$\frac{2}{3y+7} = \frac{1}{4y+1}$
14.	$\left(\frac{2}{x}+4\right)\left(\frac{x}{x+2}\right) = 7$	$\left(y^2-2\right)\left(\frac{1}{y^2-2}\right) = y$

15. Is $x^2 + 2$ a factor of $x^4 + x^3 + 2x^2 + 2x + 2$? Is $x - 3$ a factor of $x^3 - 2x^2 - 2x - 3$?

16. Simplify: a. $x^{-6} \cdot x^2$ b. $y^{-5} \div y^3$ c. $\frac{a^2 \cdot a^{-4}}{a^3}$

17. Express in scientific notation.
a. .84 b. .02 c. 17,000,000 d. .59134

The replacement set for each variable is {0, 1, 2, 3, 4}. List all the number pairs in the relations whose rules are:

a. $x + y = 4$	b. $2x = y$	c. $x = y - 1$
d. $x = \frac{1}{2}y$	e. $x = y^2$	f. $x - y = 1$
g. $2x = \frac{1}{2}y$	h. $x - 1 \leq y$	i. $4x = 3y + 1$
j. $x < y$	k. $x - y \geq 1$	l. $x^2 = y^2$

2. In set notation, state the relations whose rules are:
a. y is twice the square of x.
b. b is 1 less than twice a.
c. The sum of v and 3 is 4 less than half of s.
d. 3 less than y is greater than 2 less than z.
e. 4 times x is equal to or less than y plus 6.
f. 2 times the square of n is less than m cubed.

3. Write a rule which the number pairs in each set satisfy.
a. (1,3), (2,5), (0,1), (−2,−3), (10,21)
b. (3,9), (−3,9), (0,0), (1,1), (5,25)
c. (−3,3), (−5,5), (6,−6), (10,−10), (0,0)
d. (−3,3), (−5,5), (5,5), (3,3), (0,0)

Draw the graph of each of the following relations.

4.	$\{(x,y) \mid 2y = -x\}$	5. $\{(x,y) \mid x = -2y\}$
6.	$\{(x,y) \mid y = 2x - 2\}$	7. $\{(x,y) \mid x = 3y + 1\}$
8.	$\{(x,y) \mid 2x = -6\}$	9. $\{(x,y) \mid y = \frac{4}{3}\}$
10.	$\{(x,y) \mid y \leq 2x\}$	11. $\{(x,y) \mid y > 2x\}$
12.	$\{(x,y) \mid y \geq x + 3\}$	13. $\{(x,y) \mid y < 4\}$
14.	$\{(x,y) \mid x \geq 0\}$	15. $\{(x,y) \mid x < -1\}$
16.	$\{(x,y) \mid x = -1 \text{ or } y = -1\}$	17. $\{(x,y) \mid x = 1 \text{ and } y = 1\}$
18.	$\{(x,y) \mid y = \mid 3x \mid\}$	19. $\{(x,y) \mid y = -\mid 3x \mid\}$
20.	$\{(x,y) \mid y = \mid x \mid - 1\}$	21. $\{(x,y) \mid y = 2 - \mid x \mid\}$

22. The set {−4, −2, 0, 2, 4} is the domain of each of the relations whose rules are given below. What set is the range in each case?
a. $\{(x,y) \mid x = y\}$ b. $\{(x,y) \mid y = x + 1\}$
c. $\{(x,y) \mid x = 2y\}$ d. $\{(x,y) \mid y = 3x - 2\}$

Express the rule in each of the following functions as an equivalent rule of the form $y = mx + b$.

	a	b
1.	$\{(x,y) \mid x + y = 8\}$	$\{(x,y) \mid 2x - y = 3\}$
2.	$\{(x,y) \mid 3x + 2y = 6\}$	$\{(x,y) \mid 2x - 3y = 12\}$

3. Without drawing the graph, find the slope and y-intercept of the graph of each of the functions in **1** and **2**. Check by drawing the graph.

4. Plot $x + 2y = 3$ and $3x - y = 2$ and find the solution set common to both.

5. What are the equations, in the form $y = mx + b$, of the lines whose slopes and y-intercepts are, respectively, the following pairs.

a. 1 and 1 **b.** -1 and 3 **c.** $\frac{2}{3}$ and -1 **d.** $\frac{5}{4}$ and 0

6. Without plotting, tell which two of the following three equations have graphs that are parallel lines.

a. $3x + 2y = 6$ **b.** $3x - 2y = 6$ **c.** $y = \frac{3}{2}x - 6$

7. Which of the following relations are not functions?

a. $\{(x,y) \mid x = 3\}$ **b.** $\{(x,y) \mid y > x\}$ **c.** $\{(x,y) \mid y = 2x\}$
d. $\{(x,y) \mid x = |y|\}$ **e.** $\{(x,y) \mid x = y^2\}$ **f.** $\{(x,y) \mid 2y = x\}$

Find the slopes of the lines that contain the following pairs of points.

	a	b
8.	(0,0) and (3,5)	(0,0) and (−3,5)
9.	(−1,2) and (3,−1)	(5,0) and (0,−3)
10.	(1,−3) and (3,−3)	(2,−5) and (3,−7)

Draw the graphs of the following equations.

	a	b
11.	$x + 3y = 7$	$7x = 3y$
12.	$3x - 4y = 12$	$4y - 3x = 12$
13.	$y = 4x$	$y = -4x$
14.	$x + 5y = 0$	$x - 5y = 0$

1. By drawing graphs, find the solution set of each of the following systems. Check your solutions.

a. $\begin{cases} x - y = 9 \\ x + y = 22 \end{cases}$ **b.** $\begin{cases} x - 2y = 2 \\ 2x - 3y = 10 \end{cases}$ **c.** $\begin{cases} x + 3y = 9 \\ 2x - y = 4 \end{cases}$

d. $\begin{cases} 2x + 4y - 17 = 0 \\ x - y = 1 \end{cases}$ **e.** $\begin{cases} 3x + 4y = 45 \\ x - 2y = -5 \end{cases}$ **f.** $\begin{cases} 3x - 5y = 5 \\ -x + 2y = -1 \end{cases}$

2. Solve by addition or substitution.

a. $\begin{cases} x + y = 14 \\ x - y = -2 \end{cases}$ **b.** $\begin{cases} 2x + 3 = 14 \\ x - y = 4 \end{cases}$ **c.** $\begin{cases} 2x + 4y = 17 \\ x - \frac{1}{2}y = 1 \end{cases}$

d. $\begin{cases} 3x + 4y = 17 \\ 5x + 6y = 27 \end{cases}$ **e.** $\begin{cases} 4x + 4y = 1\frac{1}{2} \\ 2x - 4y = 0 \end{cases}$ **f.** $\begin{cases} 3x + y = 7 \\ 4x - 4y = 4 \end{cases}$

g. $\begin{cases} 8x + 12y = 37 \\ 4x - 3y = -2\frac{1}{2} \end{cases}$ **h.** $\begin{cases} 3x - 5y = 5 \\ 4y = 2x - 2 \end{cases}$ **i.** $\begin{cases} 4x + 5y = 24 \\ 3x - 2y = -5 \end{cases}$

j. $\begin{cases} 3x - 4y - .37 = 0 \\ x - y - .13 = 0 \end{cases}$ **k.** $\begin{cases} 3x + 4y = -3 \\ 5x + 5y = 5 \end{cases}$

3. Identify the equations in the following systems as consistent, inconsistent, or dependent. Solve the systems.

a. $\begin{cases} 4x + 2y = 16 \\ 2x + y = 8 \end{cases}$ **b.** $\begin{cases} 3x - 5y = 12 \\ x + y = 9 \end{cases}$ **c.** $\begin{cases} 3x - 2y = 6 \\ 3x - 4 = 2y \end{cases}$

d. $\begin{cases} .25x + .125y = .0625 \\ x + .5y = .25 \end{cases}$ **e.** $\begin{cases} 5x - 3y + 10 = 0 \\ 2.5x - 1.5y - 6 = 0 \end{cases}$

Draw the graph of the solution set of each of the following.

4. a. $x + 3y > 0$ **b.** $3x - y + 2 < 0$ **c.** $-2x - y \leq 4$
5. a. $4x - 2y < 10$ **b.** $y > 3x$ **c.** $2x + 2y \geq 20$

6. By drawing graphs, find the solution set of each of the following systems. Describe the geometric figure that is formed and find the coordinates of the vertices.

a. $\begin{cases} 3x + 4y \geq -6 \\ 2x + 2y \leq 10 \\ y \leq -2 \end{cases}$ **b.** $\begin{cases} 4x - 3y \leq 6 \\ 2x + 6y \leq 13 \\ 3x + 3y \geq -8 \end{cases}$ **c.** $\begin{cases} 8x + 4y \leq 18 \\ 4x - 2y \geq -3 \\ x \leq y \end{cases}$

d. $\begin{cases} 5x + y \leq 30 \\ x + 2y \leq 30 \\ x \geq 0 \\ y \geq 0 \end{cases}$ **e.** $\begin{cases} 2x + 4y \leq 40 \\ 5x + 2y \leq 40 \\ x \geq 0 \\ y \geq 0 \end{cases}$

7. For $9x + 4y = p$, what is the largest replacement for p that produces a graph that intersects the graph you drew for **6d**? for **6e**?

1. Write as a terminating or repeating decimal.

a. $\frac{53}{10}$ **b.** $\frac{15}{33}$ **c.** $\frac{1}{8}$ **d.** $\frac{19}{99}$ **e.** $\frac{5}{9}$ **f.** $\frac{17}{3}$ **g.** $\frac{11}{9}$ **h.** $\frac{53}{99}$ **i.** $\frac{-7}{99}$ **j.** $\frac{245}{999}$

2. Write *I* on your paper if the number is irrational. If it is rational express it as the quotient of two integers.

a. .3795 **b.** .101101101101101 . . . **c.** .212211221112 . . .
d. $.\overline{54}$ **e.** .101001000100001 . . . **f.** $2.\overline{34}$
g. $.730\overline{}$ **h.** −.34343434343434 . . . **i.** $8.320\overline{}$

3. Arrange the numbers in **2** by size, starting with the smallest.

4. Show that $\frac{4}{5}r$ is irrational when r is irrational.

5. Simplify. Then approximate each to the nearest hundredth.

a. $\sqrt{32}\cdot\sqrt{2}$ **b.** $\sqrt{300}$ **c.** $\sqrt{2}(4\sqrt{72}-3\sqrt{6})$
d. $3\sqrt{24}\div 4\sqrt{3}$ **e.** $\sqrt{6}\div\sqrt{3}$ **f.** $(\sqrt{96}-3\sqrt{24})\div\sqrt{3}$

6. Solve.

a. $\sqrt{7x-2}=5$ **b.** $\sqrt{4x-4}=8$ **c.** $7\sqrt{x-2}=5$
d. $\sqrt{3x+4}=0$ **e.** $\frac{\sqrt{4n-2}}{7}=2$ **f.** $\sqrt{\frac{3x+5}{8}}=4$
g. $2\sqrt{2r-1}=36$ **h.** $\sqrt{8x-2}=2$ **i.** $\sqrt{5x-4}+8=17$

7. Find two consecutive integers such that the square root of their sum is:

a. 5 **b.** 9 **c.** 11 **d.** 25 **e.** 19 **f.** 529

8. Find the radius of a circle whose area in square centimeters is:

a. 169π **b.** 225π **c.** $10{,}000\pi$ **d.** 3600π

9. Transform the following so that the subjects of the formulas are t, r, S, and r, respectively.

a. $S=\frac{1}{2}gt^2$ **b.** $V=Khr^2$ **c.** $V=\frac{S^2h}{3}$ **d.** $h=80-16r^2$

10. Find the radius of a cylinder whose volume in cubic feet and height in feet are, respectively:

a. 96π and 6 **b.** 125π and 5 **c.** 198π and 3

11. Find the radius of a circle (nearest thousandth) whose area in square meters is:

a. 48π **b.** 60π **c.** 90π

1. Write these disjunctions as equivalent quadratic equations.

a. $x=7$ *or* $x=-7$ **b.** $y=\frac{1}{4}$ *or* $y=-\frac{1}{4}$
c. $6z=9$ *or* $6z=-9$ **d.** $.5n=1.3$ *or* $.5n=-1.3$

Find the solution set of each of the following (approximate to the nearest hundredth when necessary).

	a	b	c
2.	$x^2=196$	$7z^2=112$	$3z^2=21$
3.	$8x^2=128$	$5z^2=125$	$(y-6)(y+6)=64$
4.	$x^2-11x-42=0$	$x^2-8x+12=0$	$x^2+x-12=0$
5.	$3x^2+14x-5=0$	$6r^2+18r=24$	$x+\frac{70}{x}=17$
6.	$(x-3)^2=25$	$(3-y)^2=9$	$(2x-1)^2=36$

Solve. (You may leave irrational answers in radical form; simplify where necessary.)

7.	$x^2-10x=144$	$x^2-10x=5$	$y^2+6y=9$
8.	$5x^2-10x-30=0$	$2x^2-7x=3$	$2x-\frac{1}{x}=3$
9.	$x^2+5=5-x$	$x^2+x-1=0$	$3x^2+5x+1=0$

Find and draw graphs of the solution sets.

10. $x^2>49$ $x^2+4x<0$ $x^2-6x\geq 0$

Find the solution set of each of the following.

11.	$y^2\leq 256$	$3x^2+5x<2$	$3z^2>5z+2$
12.	$(x+2)^2\leq 36$	$(3y-2)^2\geq 196$	$(5z+8)^2\leq 64$
13.	$x^2+4x+4>36$	$x^2+4x\geq 32$	$x^2+4x\geq 4$

Draw graphs of the solution sets.

14. a. $y=x^2-x-6$ **b.** $y=4+3x-x^2$

15. What are the range and domain of each of the functions in **14a** and **14b**?

16. Using the graphs in **14a** and **14b**, solve the following. (You may approximate to the nearest $\frac{1}{4}$ when necessary.)

a. $0=x^2-x-6$ **b.** $6=4+3x-x^2$ **c.** $-3=4+3x-x^2$

17. The product of two consecutive odd integers decreased by their sum is 47. What are the integers?

18. The length of a rectangle is 2 inches greater than its width. If the number of square inches in the area of the rectangle is equal to the number of inches in its perimeter, what is the width of the rectangle (estimate to the nearest tenth)?

In the coordinate system on a line, find the lengths of line segments with the following end points.

1. **a.** -6 and 4 **b.** -3 and -10 **c.** 0 and 12
2. **a.** -5 and 11 **b.** $4\frac{1}{4}$ and $-3\frac{1}{4}$ **c.** -6.1 and 4.9
3. **a.** 1400 and -6 **b.** -47 and 47 **c.** $3\frac{1}{3}$ and $-2\frac{1}{3}$

4. Tell whether each of the following pairs is on a line parallel to the x-axis, or to the y-axis. Give the distance between each pair of points and the midpoint of the line segment they determine.

a. $(4,2)$ and $(8,2)$ **b.** $(7,5)$ and $(7,-2)$
c. $(-4,4)$ and $(4,4)$ **d.** $(6,3)$ and $(6,-5)$
e. $(8,\frac{1}{2})$ and $(8,-5\frac{1}{2})$ **f.** $(\frac{3}{4},9)$ and $(\frac{3}{4},15)$

5. Use the distance formula to determine the distances between the following pairs of points. Simplify your solutions. Leave irrational numbers in radical form.

a. $(1,3)$ and $(5,6)$ **b.** $(2,4)$ and $(3,5)$
c. $(-1,2)$ and $(4,5)$ **d.** $(-3,-3)$ and $(-8,-3)$
e. $(5.2,3.4)$ and $(6.2,1.3)$ **f.** $(-6.3,4.4)$ and $(-2.7,-3.3)$

6. Find the slope and y-intercept of the line that contains each of the following pairs of points. Then find an equation for the line.

a. $(-2,-1)$ and $(5,3)$ **b.** $(-3,-2)$ and $(-5,-7)$
c. $(5,6)$ and $(2,0)$ **d.** $(0,0)$ and $(-2,5)$

7. Test whether the following sets of points are collinear.

a. $(1,6)$, $(-2,0)$, $(2,8)$ **b.** $(3,4)$, $(-3,-4)$, $(1\frac{1}{2},2)$

8. Find the coordinates of the midpoint of each of the segments whose coordinates are given in **6.**

9. In each case, find whether the line containing points A and B is parallel to the line containing points C and D.

A	B	C	D
$(-4,5)$	$(2,0)$	$(-2,7)$	$(3,2)$
$(-5,8)$	$(1,-4)$	$(1,8)$	$(5,0)$
$(3,11)$	$(6,8)$	$(6,-1)$	$(9,2)$

10. Suppose A, B, and C have the following coordinates. Test whether AB is perpendicular to BC in each case.

A	B	C
$(2,6)$	$(3,2)$	$(7,3)$
$(4,5)$	$(9,5)$	$(10,4)$
$(-1,3)$	$(2,5)$	$(4,2)$

1. Draw in standard position each of the angles whose measures are as follows. Label each angle.

a. $m\angle BOC = 25°$ **b.** $m\angle DOE = 42°$ **c.** $m\angle XOY = 17°$

2. Choose a point on the terminal side of each angle you drew for **1** and determine the tangent of the angle (nearest hundredth). Check your results by using the table, page 427.

3. Tell, to the nearest degree, the measure of the angle a line makes with the x-axis when it passes through the origin and the point whose coordinates are:

a. $(4,6)$ **b.** $(8,2)$ **c.** $(12,4)$ **d.** $(19,2)$
e. $(\frac{1}{2},\frac{1}{2})$ **f.** $(\frac{3}{4},\frac{1}{2})$ **g.** $(300,140)$ **h.** $(236,127)$

4. Each of the following refers to a right triangle ABC, with a the measure of the side opposite $\angle A$; b the measure of the side opposite $\angle B$; and c the measure of the hypotenuse. ($\angle C$ is the right angle.) Express answers to the nearest whole number.

a. $m\angle A = 25°$; $a = 3$ feet. Find b and c.
b. $m\angle A = 34°$; $a = 15$ feet. Find b and c.
c. $m\angle B = 81°$; $b = 26$ inches. Find a and c.
d. $m\angle B = 19°$; $b = 468$ feet. Find a and c.

5. In each of the following, find $m\angle\theta$ to the nearest degree or the value for x or y to the nearest whole number.

a. $\sin\theta = \frac{1}{2}$ **b.** $\tan\theta = \frac{8}{3}$ **c.** $\cos\theta = \frac{27}{44}$ **d.** $\sin\theta = \frac{27}{44}$
e. $\cos\theta = \frac{17}{32}$ **f.** $\cos\theta = \frac{26}{53}$ **g.** $\tan\theta = \frac{16}{41}$ **h.** $\cos\theta = \frac{241}{780}$
i. $\sin 30° = \frac{x}{15}$ **j.** $\tan 76° = \frac{100}{y}$ **k.** $\cos 15° = \frac{x}{55}$

6. Draw a vector that represents each of the following forces acting in the indicated direction.

a. 90 grams to the north **b.** 45 grams to the northeast
c. 200 pounds; N10°E **d.** 46 ounces; N30°E

7. In each case, find the resultant of the two forces.

a. a force of 40 grams acting at a heading of N40°E and a force of 60 grams acting to the east
b. a force of 30 pounds acting at a heading of N30°E and a force of 50 pounds acting to the east

1. Assume that you have a box of 8 red and 4 blue marbles. Give the measure of probability for selecting, sight unseen, on one try:

a. a red marble **b.** a blue marble **c.** a yellow marble
d. a red marble or a blue marble

2. Construct a sample space for two sets of cards. One set consists of 6 blue cards numbered 1 through 6. The other set consists of 5 red cards numbered 1 through 5. Find the probability that the sum of the number represented on a blue card and the number represented on a red card is:

a. 11 **b.** 1 **c.** 8 **d.** 3 **e.** 0 **f.** 5

3. A penny, a nickel, a dime, and a quarter are tossed together. Construct a sample space showing all of the possible outcomes.

4. Using the sample space in **3**, find the probability that:

a. The four coins show heads. **b.** There are 2 heads and 2 tails.
c. There are 3 coins showing the same side and 1 showing the other.
d. There are more tails than heads.
e. All coins fall the same way.

5. An oil prospector drills 4 wells in different locations. He knows he may either succeed in finding oil (S) or he may fail (F). Assume that S and F are equally likely. Draw a probability tree of the possible outcomes. Then find the probability that he finds oil at:

a. all locations **b.** two locations **c.** one location

Using the sample space you drew for **2**, find each of the following.

	a	b	c	d
6.	$P(r = 5)$	$P(r = b + 1)$	$P(r + b = 8)$	$P(b = 4)$
7.	$P(r + b < 7)$	$P(r > b + 2)$	$P(r \leq b + 2)$	$P(r \geq b + 2)$

	a	b
8.	$P(r = b \text{ and } r > 2)$	$P(r \leq b \text{ and } b = 5)$
9.	$P(r = b \text{ or } r = 1)$	$P(2r = b \text{ or } r < 5)$

10. Draw two disks like those on page 395. Divide the red disk into 4 equal sectors numbered 1 through 4 and the blue disk into 5 equal sectors numbered 1 through 5. Determine all possible combinations for the pair (r,b) obtained by spinning both needles and show the combinations in a sample space. Then list the truth set of:

a. $r = 5$ **b.** $r = 2$ **c.** $b = 6$ **d.** $r = 4$ *and* $b = 1$

Challenge

For use following page 41

Let us explore relationships involving the numbers that are members of the set $\{0, 1, 2, 3, \ldots\}$.

Pick any six consecutive numbers such as 7, 8, 9, 10, 11, 12. ...vide 7 by 3. Is there a remainder of 1? Divide 8 by 3. What is ...e remainder? Divide 9 by 3. Is there a remainder? Call this a ...mainder of 0. Divide 10 by 3; 11; 12. What are the remainders?

Repeat the experiment using 18, 19, 20, 21, 22, 23, 24. That is, divide each by 3 and record the remainders you obtain. How many different remainders are there? Name them.

2. Now construct three subsets of the set $\{0, 1, 2, 3, \ldots\}$ in terms of the remainders obtained when the numbers are divided by 3.

Numbers that yield a remainder of 0: $\{0, 3, 6, 9, \ldots\}$
Numbers that yield a remainder of 1: $\{1, 4, 7, 10, \ldots\}$
Numbers that yield a remainder of 2: $\{2, 5, 8, 11, \ldots\}$

Call these subsets the "0" class, the "1" class, and the "2" class, respectively. Now experiment. Add a number of the "1" class (say, 4) to a number of the "2" class (say, 11). The sum is a member of which class? Try the same experiment with another pair of numbers, one from the "1" class and the other from the "2" class. Is the sum in the "0" class? Try a few more examples.

3. Let us use the remainders 0, 1, and 2 as a means of referring to the relationship we have just discovered. Using the symbol $\oplus$ for the "addition" we have performed, we can write: $1 \oplus 2 = 0$. Where have you seen a similar sentence in Chapter 1?

4. Experiment to see whether the facts for "addition in modulo 3" (page 11) correspond to relationships in the "remainder" classes. Write three illustrations for $2 \oplus 2 = 1$; for $0 \oplus 2 = 2$; and so on.

5. If we let x and y represent any members of the set $\{0, 1, 2, 3, \ldots\}$, then $3x$ and $3y$ represent any two members of the "0" class. What will $3x + 1$ represent? $3y + 2$? $3y + 1$? $3x + 2$? Now, using these expressions involving variables, show that: the sum of any two members of the "2" class is a member of the "1" class; the sum of any member of the "1" class and any member of the "2" class is a member of the "0" class; the sum of any two members of the "0" class is also a member of the "0" class.

From your previous courses in mathematics, you are probably familiar with statements such as these:

> Unless otherwise indicated, when we speak of the factors of a number such as 18 or 365, we mean the *whole-number factors*.
> A *prime number* is a number greater than 1 whose only factors are the number itself and 1.

These statements can be extended to the set of integers. When we speak of the *factors* of an integer such as 18, we will mean the integral factors. In particular, let us agree that we mean *positive* integral factors. Then the factors of 6 will be 3 and 2, or 6 and 1 (not -3 and -2, or -6 and -1). A prime number is simply an integer greater than 1 that has no factor other than itself and 1. Are 3 and 2 prime numbers?

Can you quickly name the prime numbers less than 100? A Greek mathematician named Eratosthenes, in the third century B.C., offered a simple method (now called the *Sieve of Eratosthenes*) for locating the prime numbers less than 100. Make a list of numbers from 1 to 100. Use ten rows (1–10, 11–20, and so on).

Now go through the list crossing out those numerals that represent numbers having 2 as a factor (for instance, ~~4~~, ~~6~~, ~~8~~, . . .). What is the first numeral after 2 that has not been crossed out? Do not cross out 3 but cross out all succeeding numerals that represent numbers that have 3 as a factor (~~9~~, ~~12~~, ~~15~~, . . .). Repeat the procedure, using 5 as the factor; then repeat using 7 as the factor. Investigate further. Are all of the numbers that *have not* been eliminated prime numbers? List the prime numbers that are less than 100.

Suppose you are asked for the prime factors of 21. Are 3 and 7 the prime factors of 21? What are the prime factors of 12? of 39? of 84?

Have you noticed that you were asked to find *the* prime factors of each of the numbers named above? Disregarding the order in which you list the prime factors, do you obtain just one set of factors as the prime factors of a number? This fact—that every positive integer has just one set of prime factors—is called the **fundamental theorem of arithmetic.**

Look back at the I, J, K elastic-strip system that was introduced on page 12. You know that the operation $\uparrow$ is commutative and associative, and that the set consisting of the I, J, K strips is closed under the operation $\uparrow$. Are these three of the commutative group properties? What are the other two properties that must be satisfied for a set to be a commutative group?

1. Which (if any) element in the set $\{I, J, K\}$ can replace x and make each of these sentences true?

   a. $x \uparrow J = J$ **b.** $x \uparrow K = K$ **c.** $x \uparrow I = I$

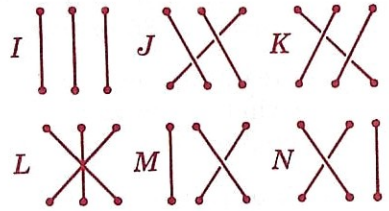
   When which element (if any) is the identity element for $\uparrow$?

2. Can you find replacements for q, r, and s that will make these sentences true?

   a. $q \uparrow K = I$ **b.** $J \uparrow r = I$ **c.** $I \uparrow s = I$

   Then does every element have an inverse element for $\uparrow$?

3. Is the set consisting of I, J, and K a commutative group under the operation of $\uparrow$?



4. Now try the following experiment. Suppose that you have a set of elastic strips consisting of six elements, as shown at the right. Let the operation be $\uparrow$, as described on page 12. The results of $L \uparrow J$ and $L \uparrow K$ have been entered in the table. Copy and complete the table. Then answer these questions:

$\uparrow$	I	J	K	L	M	N
I						
J						
K						
L		M	N			
M						
N						

Is the set $\{I, J, K, L, M, N\}$ closed under the operation $\uparrow$?

Is the operation $\uparrow$ commutative? Remember, if the operation is not commutative for even one pair of elements, then the operation is not commutative.

Is I the identity element? Does every element have an inverse element? If you test, you will find that the operation is associative. However, as you have seen, the operation $\uparrow$ is *not* commutative. Mathematicians say that the set $\{I, J, K, L, M, N\}$ is a non-commutative group under the operation $\uparrow$. If any of the other properties were not satisfied, the set would not be a group.

Puzzle Problems

1. David and his father agreed that whenever David's mark in his mathematics test was 85 or more, David would receive 50 cents, but if it were less than 85, David would forfeit 25 cents. During the year there were 20 tests, and David's "Math Test" account showed a balance of \$4.75. How many of his tests showed a mark of 85 or more? (Hint: Let x represent the number of test marks less than 85. Represent the number of test marks showing 85 or more. Represent the amount David gained; the amount he lost.)

2. The next year David and his father decided to change the agreement as follows: for every mark of 85 or more, David was to receive 50 cents; for every mark of 70 or less, David was to forfeit 40 cents; for every mark between 70 and 85, he would neither receive nor forfeit anything. For this year David's "Math Test" account showed a balance of \$5.40. There were 20 tests in the year, and David had twice as many marks in the "70 or less" category as he had in the "between 70 and 85" category. How many of his tests showed a mark of 85 or more?

3. Five years ago Bob was three times as old as Larry. In five years he will be twice as old as Larry. How old is each now? (Hint: Let x represent Larry's age five years ago.)

4. Mr. Brown had some marbles. He gave half of them to his oldest grandson, one-third of them to the next oldest, one-eighth to the third, and the marble that was left over he gave to his youngest grandson. How many marbles were there in all?

5. "Take a number," Tom said to Sally. "Double it, add 3, divide by 5. What is your result?"

Sally answered: "If I add 3 to twice my result I get the number I chose."

What is the number that Sally chose?

6. Jim said to Harold: "Find the value of $3x + 2$ when x is replaced by 2 and then find the value of $3x + 2$ when x is replaced by this value." Harold computed: $3 \cdot 2 + 2 = 8$; then $3 \cdot 8 + 2 = 26$.

Harold then countered with this question: "What would the first x-replacement be had I ended with 44?"

Can you answer the question? (Of course, you can find the answer by guessing, but try to do it by writing an equation and solving it.)

Arithmetic Series

Study this ordered set of numbers $\{2, 7, 12, 17, \ldots, 42\}$. Name the four numbers that follow 17 in this ordered set. It is characteristic of such a set that the difference between each number (except the first) and the preceding number is the same. We call this difference the **common difference.** Does the ordered set $\{2, 5, 8, 11, 14\}$ have a common difference? What is it?

1. We shall find the sum of the numbers in an ordered set when it has a common difference. We illustrate the method by an example, first, using $\{2, 5, 8, 11, 14\}$. Let S represent the sum.

We use the associative and commutative principles to re-express S as shown in line (2) at the right. Explain how line (3) follows from lines (1) and (2). The sentence in line (3) may be written $2S = 5 \cdot 16$, or its equivalent $S = \frac{5}{2} \cdot 16$, or 40.

(1) $S = 2 + 5 + 8 + 11 + 14$

(2) $S = 14 + 11 + 8 + 5 + 2$

(3) $2S = 16 + 16 + 16 + 16 + 16$

Carry out this line of reasoning with the numbers in the set $\{2, 5, 8, \ldots, 23\}$.

$$S = 2 + 5 + 8 + 11 + 14 + 17 + 20 + 23$$

Do you find that $S = \frac{8}{2}(2 + 23)$, or $4 \cdot 25$, or 100?

2. Let a represent the first number and l the last in an ordered set of numbers that have d as a common difference. Can the second number in the set be represented as $a + d$? Can the next-to-last number in the set be represented as $l - d$? Find the sum of the second number and the next-to-last number. Is the sum $a + l$?

3. Can the number that follows $a + d$ be represented as $a + 2d$? How would you represent the number before the next to the last? What is the sum of the two numbers you have just represented? Can we say that:

$$S = a + (a + d) + (a + 2d) + \ldots + (l - 2d) + (l - d) + l$$

Now show that: $2S = n(a + l)$

and: $S = \frac{n}{2}(a + l)$

This is a formula for the sum of an **arithmetic series** of n terms, with a the first term and l the last term.

4. Using this formula, find: the sum of the integers between 1 and 100, inclusive; the sum of the multiples of 3, from 3 to 300, inclusive, that is, $3 + 6 + 9 + \ldots + 300$; the sum of the integers from -50 to 75, inclusive.

In Chapter 1 (pages 37–40), you proved that the sum of two even numbers is an even number, the sum of two odd numbers is an even number, and the sum of an odd number and an even number is an odd number. Using your knowledge of the multiplication of polynomials, you can now investigate products of odd and even numbers. Again, let us agree that the replacement set for the variables x and y is the set $\{0, 1, 2, 3, \ldots\}$.

1. First, investigate the product of any two even numbers. Do $2x$ and $2y$ represent any two even numbers? Then $2x \cdot 2y$, or $4xy$, represents the product of any two even numbers. Is $4xy$ an even number, or is it an odd number? Give your reasoning.

2. Consider the product of two odd numbers. Can we represent these numbers as $2x + 1$ and $2y + 1$? Then: $(2x + 1)(2y + 1) = 4xy + 2x + 2y + 1$. Show that $4xy + 2x + 2y + 1$ must represent an odd number. Then is the product of any two odd numbers an odd number?

3. The final case to be considered is the product of an even number and an odd number. Does $2x \cdot (2y + 1)$ represent the indicated product of any even number and any odd number? Can we write the product as $2 \cdot (2xy + x)$? Is the product of an even number and an odd number an odd number, or is it an even number?

⊕	E	O
E	E	O
O	O	E

⊗	E	O
E	E	E
O	E	O

4. At the left are tables that use the symbol E to represent *any* even number and the symbol O to represent *any* odd number. Are the entries in the table consistent with your discoveries in Chapter 1 and on this page, if ⊕ is interpreted as addition and ⊗ is interpreted as multiplication?

5. Think now of a set consisting of two elements, E and O, with the operations ⊕ and ⊗ as defined by the table.

Using the table for ⊕, consider each property that must be satisfied for a set to be a commutative group. Is the set $\{E, O\}$ a commutative group under ⊕? (What is the identity element for ⊕? Does each element have an inverse element for ⊕?)

Using the table for ⊗, go over the commutative group properties. Is the set $\{E, O\}$ a commutative group under the operation ⊗? If your answer is "Yes," then recheck your work. Is O the identity element for ⊗? Does each element have an inverse element for ⊗?

1. Let us take another look at the procedure of division as illustrated in the example on page 176 (repeated below). Is x the only variable involved? Is the variable always written in decreasing powers of x? We can then omit the x's and write:

$$\begin{array}{r} x + 2 \\ x + 1 \overline{)\, x^2 + 3x + 2} \\ \underline{x^2 + 1x} \\ 2x + 2 \\ \underline{2x + 2} \\ 0 \end{array}$$

$$\begin{array}{r} 1 + 2 \\ 1 + 1 \overline{)\, 1 + 3 + 2} \\ \underline{1 + 1} \\ 2 + 2 \\ \underline{2 + 2} \\ 0 \end{array}$$

Inspect these numbers. Are there always some numbers which are duplicated? To save time, we will not write any of the duplications:

$$\begin{array}{r} \\ 1 + 1 \overline{)\, 1 + 3 + 2} \\ \underline{\quad 1} \\ 2 \\ \underline{\quad\quad 2} \\ 0 \end{array}$$

To save space we "compress" to:

$$\begin{array}{r} \\ 1 + 1 \overline{)\, 1 + 3 + 2} \\ \underline{\quad 1 + 2} \\ 2 + 0 \end{array}$$

If we always divide by a binomial of the form $x + r$, we can drop the 1 coefficient in the divisor; and since addition is easier than subtraction, we can use $-r$ instead of r and then add the partial products. Furthermore, since the coefficient of the first term in the quotient is the same as that of the dividend, we can write the example as shown at the right. What is the remainder?

$$\begin{array}{r} \\ \underline{-1)}\ 1 + 3 + 2 \\ \underline{\quad - 1 - 2} \\ 1 + 2 + 0 \end{array}$$

2. This procedure is known as *synthetic division*. For a further illustration, divide $2x^3 - x^2 - 7x - 4$ by $x + 2$ using "long division." Then compare your work with the synthetic division shown here.

$$\begin{array}{r} \\ \underline{-2)}\ 2 - 1 - 7 - 4 \\ \underline{\quad - 4 + 10 - 6} \\ 2 - 5 + 3 - 10 \end{array}$$

$2x^2 - 5x + 3$ (quotient)

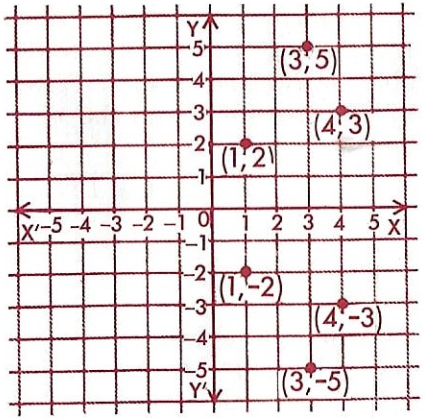
-10 (remainder)

3. Divide, using synthetic division. Include all coefficients of terms (even 0 coefficients). For example, the dividend $3x^3 + x + 2$ would be written as $3 + 0 + 1 + 2$.

a. $(2x^2 + 5x - 26) \div (x - 2)$ **b.** $(x^3 + 4x^2 - 7x - 30) \div (x + 3)$

c. $(x^4 + 4x^3 + 27) \div (x + 3)$ **d.** $(3y^3 - 8y^2 - 40y + 25) \div (y - 5)$

Consider the set of points $\{(1,2), (3,5), (4,3)\}$. A second set may be derived from these by taking the opposite of the y-coordinates and obtaining $\{(1,-2), (3,-5), (4,-3)\}$. If we think of the x-axis as a "mirror," the points in the second set are "mirror images" of the corresponding points in the first set. We call $(1,-2)$ a *reflection* of $(1,2)$ *in the x-axis*. What is the reflection of $(3,5)$ in the x-axis? of $(4,3)$? Two points are reflections of each other in a line if the line is the perpendicular bisector of the line segment joining the two points. The line is called an *axis of reflection.*



1. What is the reflection of $(1,2)$ in the y-axis? of $(3,5)$? of $(4,3)$?

2. Consider $A(2,4)$. What are the coordinates of B, the reflection of A in the x-axis? What are the coordinates of C, the reflection of B in the y-axis? How can we find the coordinates of C directly from those of A?

3. What is the reflection of $(3,6)$ in the line whose equation is $y = 3$? $y = 8$? $x = 1$? $x = 5$?

4. What is the reflection of $(3,5)$ in the line whose equation is $y = x$?

5. Find the reflections of each of the following points in the line $y = x$: $(2,4)$, $(0,3)$, $(-3,2)$, $(2,-3)$, $(3,3)$, $(0,0)$. State a rule for finding the reflection of any point in the line $y = x$.

6. Two figures are reflections of each other in a line if each point in one figure is a reflection of a point in the other figure. A triangle has vertices at $A(0,2)$, $B(5,1)$ and $C(2,4)$. What are the coordinates of the vertices of the triangle that is the reflection of triangle ABC in the x-axis? in the y-axis? How do these reflections compare with each other?

1. Consider the relation whose rule is $y = 2^x$ (that is, y equals 2 raised to the xth power) where the domain of x is the set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

When x is replaced by -3, y must equal 2^{-3}; that is, $\frac{1}{2^3}$, or $\frac{1}{8}$. Thus, $(-3,\frac{1}{8})$ is a member of the relation.

When x is replaced by -2, y must equal 2^{-2}; that is, $\frac{1}{2^2}$, or $\frac{1}{4}$. Thus, $(-2,\frac{1}{4})$ is a member of the relation.

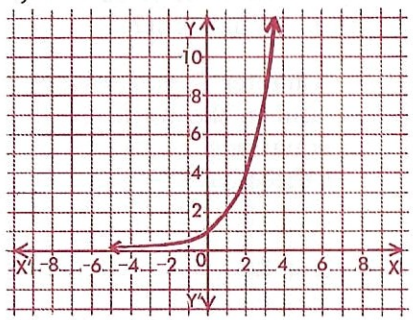
Find the number pair that is a member of the relation when x is replaced by each of the numbers indicated in the table below.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	?	?	$\frac{1}{8}$	$\frac{1}{4}$	?	?	?	?	?	?	?

2. For each replacement of x, do we obtain just one value of y? Then is the relation $\{(x,y) \mid y = 2^x\}$ a function?

3. Is the value of y ever negative?

4. On a set of coordinate axes, plot the number pairs you obtained in **1.** Do they seem to lie on a straight line? If you drew a smooth curve through your points, would it resemble the curve shown in this figure? Of course, it would not be correct for you to draw this curve, since the domain of x was given as the set of integers.



Can you give a reason why the domain was limited to the integers? It is because in your work the definition of *exponent* limits exponents to members of the set of integers. But in a future mathematics course, you will find that the definition of exponents can be extended to include all positive and negative numbers as exponents. In such a case, the domain of x can meaningfully include zero and all positive and negative numbers; and the graph of the function whose rule is $y = 2^x$ will be the curve shown above. We say that this is an example of an **exponential function.**

5. Using the set $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ as the domain in each case, make a table of values and plot the graph of each of the following functions.

a. $\{(x,y) \mid y = 3^x\}$ **b.** $\{(x,y) \mid y = 4^x\}$ **c.** $\{(x,y) \mid y = (\frac{1}{2})^x\}$

1. Apply addition-subtraction or substitution methods of solution to the following systems.

a. $\begin{cases} 5x - 3y = -7 \\ -2x + 9y = 4 \\ x + 2y = -1 \end{cases}$

b. $\begin{cases} 2x - y - 1 = 0 \\ x - 2y + 1 = 0 \\ x + y + 1 = 0 \end{cases}$

2. Would you call **1a** a system of consistent equations? Would you call **1b** consistent? inconsistent? dependent? Sketch the graphs of the two systems of **1**. Systems such as that in **1b** are sometimes called *indeterminate*. Any two of the three equations can have a common solution but the three equations do not have a point in common on their graphs. Neither are they parallel or coincident lines.

3. Consider the system at the right. Determine the value of K so that the system has a solution. What is it?

$$\begin{cases} 2x - y - 1 = 0 \\ x - 2y + 1 = 0 \\ x + y + K = 0 \end{cases}$$

4. In Chapter 7, you solved systems of two equations in two variables; and in the work on this page, you have solved systems with three equations in two variables. Now consider the following system of three equations in three variables.

(1) $x + 2y + z = 0$
(2) $x - y - z = 2$
(3) $3x + y + 2z = -1$

Is it possible to form a new equation (1′) from (1) and (2) which does not contain x? Is it possible to form a new equation (3′) from (2) and (3)? These two new equations form a new system. Solve this system. Use the solution set as replacements for y and z in the original system. What is the solution of the original system?

(1′) $3y + 2z = -2$
(3′) $4y + 5z = -7$

5. Solve the following systems of equations.

a. $\begin{cases} 2x + y - z - 5 = 0 \\ x - 3y + 3z + 1 = 0 \\ 5x - 5y - 2z + 2 = 0 \end{cases}$

b. $\begin{cases} 4x + 5z = 6 \\ y - 6z = -2 \\ 3x + 4z = 3 \end{cases}$

c. $\begin{cases} x + z = 4y \\ x + y = 6z \\ y + z = 8x \end{cases}$

d. $\begin{cases} x + y - 2z + w = 4 \\ 5y - 7z + 2w = 1 \\ 2x - 3w + 5 = 0 \\ y + 3z + 1 = 0 \end{cases}$

1. Let us see whether we can prove that $\sqrt{2}$ is an irrational number. We begin by considering a property of integers. You know that any integer can be expressed uniquely as the product of prime factors, for example, $30 = 2 \times 3 \times 5$. If you square 30, you obtain $30 \times 30 = (2 \times 3 \times 5) \times (2 \times 3 \times 5)$. How many times does the factor 2 appear in the square of 30? How many times does each of the other factors appear in the square of 30? Suppose you had started with a number, such as 12, that has two identical factors $(12 = 2 \times 2 \times 3)$. Since 2 appears as a factor twice in 12, how often would it appear in the square of 12? In general, if a prime number appears n times in a certain number, how often will it appear in the square of that number? Is $2n$ an even number (when n is an integer)?

Do you agree that if a number is the square of some integer its prime factors must each occur an even number of times?

2. We are now ready to prove that $\sqrt{2}$ is an irrational number. To do this, we will begin by assuming that $\sqrt{2}$ is a rational number and show that this assumption leads to a contradiction.

Assume that $\sqrt{2} = \frac{P}{Q}$, where P and Q are integers and $Q \neq 0$

Then: $(\sqrt{2})^2 = \left(\frac{P}{Q}\right)^2$

and $2 = \frac{P^2}{Q^2}$ [Why is $(\sqrt{2})^2 = 2$?]

Therefore, $2Q^2 = P^2$

Is P^2 the square of an integer? Then must $2Q^2$ be the square of the same integer as P^2? But we also know that Q^2 is the square of an integer. So both $2Q^2$ and Q^2 are perfect squares.

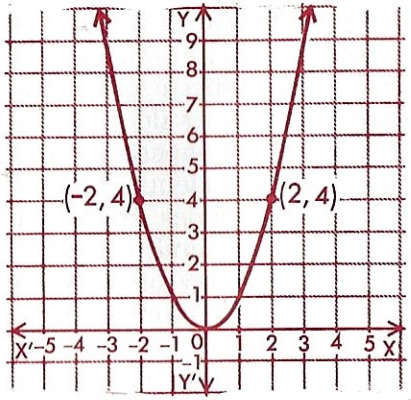
Since $2Q^2$ is a perfect square, each of its factors must appear an even number of times; therefore, 2 appears as a factor of $2Q^2$ an *even* number of times. But 2 must appear as a factor of Q^2 one less time than it appears as a factor of $2Q^2$; therefore, must it appear as a factor of Q^2 an *odd* number of times?

But Q^2 is the square of an integer, and therefore, if 2 is a factor of Q^2, it must appear as a factor an even number of times.

So we now know that 2 must appear as a factor of Q^2 both an even and an odd number of times if $\sqrt{2}$ is a rational number. Thus, we reach a contradiction by assuming that $\sqrt{2}$ is a rational number. Then we conclude that $\sqrt{2}$ is an irrational number.

3. Prove that $\sqrt{3}$ is an irrational number.

If you have not previously worked the Challenge page on reflections (page 418), do it before beginning this work.



1. The graph of $\{(x,y) \mid y = x^2\}$ is the parabola shown at the left. Is $(2,4)$ a point of the graph? Is $(-2,4)$? Is $(-2,4)$ a reflection of $(2,4)$ in the y-axis? Is $(3,9)$ on the graph? What is its reflection in the y-axis? Is this reflection also on the graph? Do you think that the reflection of every point of this graph in the y-axis is also on the graph? Try several points. The y-axis is called an *axis of symmetry* and the graph is *symmetric.*

2. Draw the graph of $\{(x,y) \mid y = x^2 - 2x\}$ in the domain $-2 \leq x \leq 4$. Is $(-2,8)$ on the graph? Is $(4,8)$? Are these points reflections of each other in the line $x = 1$? Are $(-1,3)$ and $(3,3)$ on the graph? Are they reflections of each other in the line $x = 1$? Do you think that the graph of $\{(x,y) \mid y = x^2 - 2x\}$ is symmetric? Is its axis of symmetry the line $x = 1$? If so, what should $(0,0)$ correspond to? Which point corresponds to itself?

3. Draw the graph of $\{(x,y) \mid y = x^2 + 4x - 1\}$. Is it symmetric? Does it have the line $x = -2$ as its axis of symmetry?

4. Study this table of results found in **1–3:**

Rule of Relation	Axis of Symmetry
$y = x^2$	$x = 0$
$y = x^2 - 2x$	$x = 1$
$y = x^2 + 4x - 1$	$x = -2$

Is the axis of symmetry $x = -\frac{b}{2}$ where b is the coefficient of x in the rule $y = x^2 + bx + c$?

What is the axis of symmetry of $y = x^2 + 2x$? of $y = x^2 - 6x + 3$? of $y = x^2 + 5x - 2$?

Draw the graph of each of the following. First find its axis of symmetry if you can.

5. a. $\{(x,y) \mid x = y^2\}$ **b.** $\{(x,y) \mid x = y^2 + 2y\}$

6. a. $\{(x,y) \mid x = y^2 - 2y + 1\}$ **b.** $\{(x,y) \mid x = y^2 + 3y - 1\}$

You have found that the graph of the equation $x^2 + y^2 = 36$ is a circle. Do you think that the graph of the equation $x^2 + 4y^2 = 36$ is also a circle? Let us investigate this equation and its solution set.

1. When x is replaced by 0, $4y^2$ must equal 36 and y^2 must equal 9. Then will the number pair $(0,-3)$ satisfy $x^2 + 4y^2 = 36$? Will the number pair $(0,3)$ also satisfy $x^2 + 4y^2 = 36$?

2. When x is replaced by 2, then $4y^2$ must equal 32 and y^2 must equal 8. Are $(2,\sqrt{8})$ and $(2,-\sqrt{8})$ two more number pairs in the solution set? We might, of course, express $\sqrt{8}$ as $2\sqrt{2}$; but for our purposes, it is just as well to find approximations of $\sqrt{8}$ and $-\sqrt{8}$ to the nearest tenth.

3. When x is replaced by 4, what number pair or pairs do you obtain as members of the solution set? Again find an approximation of any irrational number you may obtain.

4. When x is replaced by 6, what number pair or pairs do you obtain?

5. Find the number pair or pairs when x is replaced by -2; by -4; by -6.

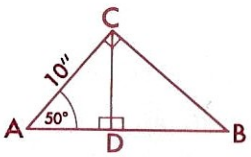
6. You now have twelve number pairs that are members of the solution set of $x^2 + 4y^2 = 36$. Plot these number pairs as points in a coordinate plane. Do the points suggest a circle? Do they suggest a curve that resembles a circle but is somewhat "flattened out"? Draw the curve. This curve is called an **ellipse.**

7. Do you see why we used numbers between -6 and 6, inclusive, as replacements for x? If we had used 7 (or -7), then x^2 would have been 49 and $4y^2$ would have had to equal -13. Will any real number, used as a replacement for y, make $4y^2$ equal to -13?

8. Suppose we think of the equation $x^2 + 4y^2 = 36$ as the rule for a relation. That is, suppose we think of the relation $\{(x,y) \mid x^2 + 4y^2 = 36\}$. Is the domain of x the set of all real numbers between -6 and 6, inclusive? What is the range of the relation?

9. Draw the graph of the relation $\{(x,y) \mid 4x^2 + y^2 = 36\}$. What is the domain of this relation? the range?

10. Draw the graph of the equation $4x^2 + 9y^2 = 144$.



1. Triangle ABC is a right triangle with its right angle at C. Segment CD is an altitude to the hypotenuse AB.

Suppose $m\angle A = 50°$. Is triangle ADC a right triangle? Is triangle CDB? Find $m\angle ACD$; $m\angle DCB$; $m\angle B$. You will find these angle measures useful in the following exercises.

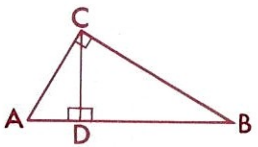
2. Suppose $mAC = 10$ inches. Which trigonometric function can you use to find the length of CD? of AD? Find each to the nearest tenth of an inch.

3. You now know mCD and $m\angle DCB$. Find mDB and mBC to the nearest tenth. Also find mAB.

We see, then, that knowing the measure of one angle ($\angle A$) and the measure of one side (AC) we are able to compute the measures of all other angles and sides in the figure.

4. Suppose in triangle ABC above, $m\angle A = 55°$ and $mBC = 100$ units. Find the measure of each angle to the nearest degree and each side to the nearest unit.

5. Using the results in **1–3**, verify that $mDB : mCD = mCD : mAD$. Verify this proportion using the results in **4**. The verification will not be exact, but close enough to suggest that the proportion is true.



6. Let us try to prove that this proportion is true for any right triangle.

a. Consider right triangles ACB and CDB. You know that $\angle ACB$ and $\angle CDB$ are equal in measure. Why? Is $\angle B$ an angle common to both triangles? Then must $m\angle A = m\angle DCB$?

b. Explain why $\tan \angle DCB = \frac{mDB}{mCD}$.

c. Explain why $\tan A = \frac{mCD}{mAD}$.

d. Explain why $\tan A = \tan \angle DCB$.

e. Why does $mDB : mCD = mCD : mAD$?

7. Using trigonometric functions, prove:

$$mAD : mAC = mAC : mAB$$

1. Arnold, Ben, Charles, and Dan went to the theater. Their seats were next to each other. To find the probability that Ben and Charles sat next to each other, we should first find the sample space of seating arrangements. One point in this sample space is (A, C, B, D) which shows that Arnold is seated at the left, then Charles, then Ben, and then Dan. List all the points in this sample space. Are there 24 such points? Using a tree, prove that there are no more than 24.

2. Find the probabilities of the following events.

a. Ben and Charles are sitting next to each other.

b. Ben and Charles are separated.

c. Charles is sitting between Arnold and Ben.

d. Charles is not sitting between Arnold and Ben.

e. Arnold and Dan are sitting in the end positions.

f. Arnold is sitting in an end position.

3. After the theater they went to a restaurant where they were seated at a round table with four chairs.

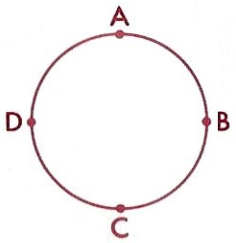


Figure 1

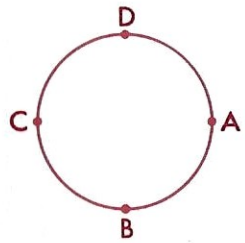


Figure 2

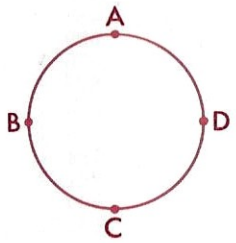


Figure 3

Let us consider the seating arrangements in Figures 1 and 2 to be the same since the relative positions are not different. (Ben is seated next to, and at the left of, Arnold; Charles is seated next to, and at the left of Ben; and so on.) But the arrangements in Figures 1 and 3 are different. Make a sample space of the seating arrangements. Do you have 6 points in the space?

4. For the seating arrangements at the round table, answer the questions in **2a–d**, inclusive.

5. Suppose that Arnold, Beatrice, Carl, and Dorothy went to the theater together. What is the probability that each boy sits next to a girl if they scramble the tickets and each chooses one at random?

6. Find the probability that each boy sits next to a girl at a round table for four, if seats are chosen at random.

N	N^2	$\sqrt{N}$	N	N^2	$\sqrt{N}$	N	N^2	$\sqrt{N}$
1	1	1.00	51	2,601	7.14	101	10,201	10.05
2	4	1.41	52	2,704	7.21	102	10,404	10.10
3	9	1.73	53	2,809	7.28	103	10,609	10.15
4	16	2.00	54	2,916	7.35	104	10,816	10.20
5	25	2.24	55	3,025	7.42	105	11,025	10.25
6	36	2.45	56	3,136	7.48	106	11,236	10.30
7	49	2.65	57	3,249	7.55	107	11,449	10.34
8	64	2.83	58	3,364	7.62	108	11,664	10.39
9	81	3.00	59	3,481	7.68	109	11,881	10.44
10	100	3.16	60	3,600	7.75	110	12,100	10.49
11	121	3.32	61	3,721	7.81	111	12,321	10.54
12	144	3.46	62	3,844	7.87	112	12,544	10.58
13	169	3.61	63	3,969	7.94	113	12,769	10.63
14	196	3.74	64	4,096	8.00	114	12,996	10.68
15	225	3.87	65	4,225	8.06	115	13,225	10.72
16	256	4.00	66	4,356	8.12	116	13,456	10.77
17	289	4.12	67	4,489	8.19	117	13,689	10.82
18	324	4.24	68	4,624	8.25	118	13,924	10.86
19	361	4.36	69	4,761	8.31	119	14,161	10.91
20	400	4.47	70	4,900	8.37	120	14,400	10.95
21	441	4.58	71	5,041	8.43	121	14,641	11.00
22	484	4.69	72	5,184	8.49	122	14,884	11.05
23	529	4.80	73	5,329	8.54	123	15,129	11.09
24	576	4.90	74	5,476	8.60	124	15,376	11.14
25	625	5.00	75	5,625	8.66	125	15,625	11.18
26	676	5.10	76	5,776	8.72	126	15,876	11.22
27	729	5.20	77	5,929	8.77	127	16,129	11.27
28	784	5.29	78	6,084	8.83	128	16,384	11.31
29	841	5.39	79	6,241	8.89	129	16,641	11.36
30	900	5.48	80	6,400	8.94	130	16,900	11.40
31	961	5.57	81	6,561	9.00	131	17,161	11.45
32	1,024	5.66	82	6,724	9.06	132	17,424	11.49
33	1,089	5.74	83	6,889	9.11	133	17,689	11.53
34	1,156	5.83	84	7,056	9.17	134	17,956	11.58
35	1,225	5.92	85	7,225	9.22	135	18,225	11.62
36	1,296	6.00	86	7,396	9.27	136	18,496	11.66
37	1,369	6.08	87	7,569	9.33	137	18,769	11.70
38	1,444	6.16	88	7,744	9.38	138	19,044	11.75
39	1,521	6.24	89	7,921	9.43	139	19,321	11.79
40	1,600	6.32	90	8,100	9.49	140	19,600	11.83
41	1,681	6.40	91	8,281	9.54	141	19,881	11.87
42	1,764	6.48	92	8,464	9.59	142	20,164	11.92
43	1,849	6.56	93	8,649	9.64	143	20,449	11.96
44	1,936	6.63	94	8,836	9.70	144	20,736	12.00
45	2,025	6.71	95	9,025	9.75	145	21,025	12.04
46	2,116	6.78	96	9,216	9.80	146	21,316	12.08
47	2,209	6.86	97	9,409	9.85	147	21,609	12.12
48	2,304	6.93	98	9,604	9.90	148	21,904	12.17
49	2,401	7.00	99	9,801	9.95	149	22,201	12.21
50	2,500	7.07	100	10,000	10.00	150	22,500	12.25

ANGLE	SIN	COS	TAN	ANGLE	SIN	COS	TAN
1°	.0175	.9998	.0175	45°	.7071	.7071	1.0000
2°	.0349	.9994	.0349	46°	.7193	.6947	1.0355
3°	.0523	.9986	.0524	47°	.7314	.6820	1.0724
4°	.0698	.9976	.0699	48°	.7431	.6691	1.1106
5°	.0872	.9962	.0875	49°	.7547	.6561	1.1504
				50°	.7660	.6428	1.1918
6°	.1045	.9945	.1051	51°	.7771	.6293	1.2349
7°	.1219	.9925	.1228	52°	.7880	.6157	1.2799
8°	.1392	.9903	.1405	53°	.7986	.6018	1.3270
9°	.1564	.9877	.1584	54°	.8090	.5878	1.3764
10°	.1736	.9848	.1763	55°	.8192	.5736	1.4281
11°	.1908	.9816	.1944	56°	.8290	.5592	1.4826
12°	.2079	.9781	.2126	57°	.8387	.5446	1.5399
13°	.2250	.9744	.2309	58°	.8480	.5299	1.6003
14°	.2419	.9703	.2493	59°	.8572	.5150	1.6643
15°	.2588	.9659	.2679	60°	.8660	.5000	1.7321
16°	.2756	.9613	.2867	61°	.8746	.4848	1.8040
17°	.2924	.9563	.3057	62°	.8829	.4695	1.8807
18°	.3090	.9511	.3249	63°	.8910	.4540	1.9626
19°	.3256	.9455	.3443	64°	.8988	.4384	2.0503
20°	.3420	.9397	.3640	65°	.9063	.4226	2.1445
21°	.3584	.9336	.3839	66°	.9135	.4067	2.2460
22°	.3746	.9272	.4040	67°	.9205	.3907	2.3559
23°	.3907	.9205	.4245	68°	.9272	.3746	2.4751
24°	.4067	.9135	.4452	69°	.9336	.3584	2.6051
25°	.4226	.9063	.4663	70°	.9397	.3420	2.7475
26°	.4384	.8988	.4877	71°	.9455	.3256	2.9042
27°	.4540	.8910	.5095	72°	.9511	.3090	3.0777
28°	.4695	.8829	.5317	73°	.9563	.2924	3.2709
29°	.4848	.8746	.5543	74°	.9613	.2756	3.4874
30°	.5000	.8660	.5774	75°	.9659	.2588	3.7321
31°	.5150	.8572	.6009	76°	.9703	.2419	4.0108
32°	.5299	.8480	.6249	77°	.9744	.2250	4.3315
33°	.5446	.8387	.6494	78°	.9781	.2079	4.7046
34°	.5592	.8290	.6745	79°	.9816	.1908	5.1446
35°	.5736	.8192	.7002	80°	.9848	.1736	5.6713
36°	.5878	.8090	.7265	81°	.9877	.1564	6.3138
37°	.6018	.7986	.7536	82°	.9903	.1392	7.1154
38°	.6157	.7880	.7813	83°	.9925	.1219	8.1443
39°	.6293	.7771	.8098	84°	.9945	.1045	9.5144
40°	.6428	.7660	.8391	85°	.9962	.0872	11.4301
41°	.6561	.7547	.8693	86°	.9976	.0698	14.3007
42°	.6691	.7431	.9004	87°	.9986	.0523	19.0811
43°	.6820	.7314	.9325	88°	.9994	.0349	28.6363
44°	.6947	.7193	.9657	89°	.9998	.0175	57.2900

ROSSKOPF · MORTON · HOOTEN
SITOMER · WILLOUGHBY · GILBERT

MODERN MATHEMATICS

ALGEBRA ONE

SILVER BURDETT COMPANY

08 10J 9